Department of Engg. Mathematics

Course : Engineering Mathematics-II (17MAT21).

Sem.: 2^{nd}

DIFFERENTIAL EQUATION-II

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Solution of second order linear differential equation with constant coefficients:

Linear differential equation of nth order with constant coefficients is defined as:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + a_n y = X$$

Where a0 , a1, a2,…,an are constants and X is a function of x.

General solution = complementary function + particular integral

Case (i) **If m_1, m_2, m_3, …,mn are real & distinct then**

C.F = $c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$

Case (ii) **If two roots are equal i.e., $m_1 = m_2 = m$**

C.F = $(c_1 x + c_2) e^{mx} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$

Case (iii) **If three roots are equal i.e., $m_1 = m_2 = m_3 = m$**

C.F = $(c_1 x^2 + c_2 x + c_3) e^{mx} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$

Case(iv) **If two roots are complex i.e.,** $m_1 = \alpha + i\beta$ $\quad m_2 = \alpha - i\beta$

C.F = $(c_1 \cos \beta x + c_2 \sin \beta x) e^{\alpha x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$

Case(v) **If two pairs of complex roots are equal** i.e.,

$m_1 = m_3 = \alpha + i\beta \quad m_2 = m_4 = \alpha - i\beta$

C.F = $((c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x) e^{\alpha x} + c_5 e^{m_5 x} + \ldots + c_n e^{m_n x}$

Problem 1 Solve the equation $\left(D^2 - D + 1\right) y = 0$

Solution:

The A.E is $m^2 - m + 1 = 0 \Rightarrow m = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1 \pm \sqrt{3}i}{2}$.

$$m = \frac{1 \pm \sqrt{3}i}{2} \text{ and } \alpha = \frac{1}{2};\ \beta = \frac{\sqrt{3}}{2}$$

$$G.S: y = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)$$

$$G.S: y = e^{\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}x}{2}\right)$$ where A, B are arbitrary constants.

$$5y = 5Ae^{-2t} + 5Be^{-7t} + \frac{5e^{2t}}{6} - \frac{5t}{7} + \frac{45}{98}$$

$$(D+5)y = 3Ae^{-2t} - 2Be^{-7t} + \frac{7e^{2t}}{6} - \frac{5t}{7} - \frac{1}{7} + \frac{45}{98}$$

$$(2) \Rightarrow 2x = -(D+5)y + e^{2t}$$

$$= -3Ae^{-2t} + 2Be^{-7t} - \frac{7e^{2t}}{6} + \frac{5t}{7} - \frac{31}{98} + e^{2t}$$

$$x = \frac{-3A}{2}e^{-2t} + Be^{-7t} - \frac{7}{72}e^{2t} + \frac{5t}{14} - \frac{31}{196}$$

The General solution is

$$x = \frac{-3A}{2}e^{-2t} + Be^{-7t} - \frac{e^{2t}}{12} + \frac{5t}{14} - \frac{31}{196}$$

$$y = Ae^{-2t} + Be^{-7t} + \frac{e^{2t}}{6} - \frac{t}{7} + \frac{9}{98}.$$

Inverse Differential Operator And Particular Integral

Consider a differential equation

$$f(D)\, y = x \qquad ...(1)$$

Define $\frac{1}{f(D)}$ such that

$$f(D)\left\{\frac{1}{f(D)}\right\} x = x \qquad ...(2)$$

Here $f(D)$ is called the inverse differential operator. Hence from Eqn. (1), we obtain

$$y = \frac{1}{f(D)} x \qquad ...(3)$$

Since this satisfies the Eqn. (1) hence the particular integral of Eqn. (1) is given by Eqn. (3)

Thus, particular Integral (P.I.) = $\frac{1}{f(D)} x$

The inverse differential operator $\frac{1}{f(D)}$ is linear.

i.e., $$\frac{1}{f(D)}\{ax_1 + bx_2\} = a\frac{1}{f(D)}x_1 + b\frac{1}{f(D)}x_2$$

where a, b are constants and x_1 and x_2 are some functions of x.

Problem 8 Solve $\left(D^2+a^2\right)y=\sec ax$.

Solution:

The A.E. is $m^2+a^2=0$

$\Rightarrow m^2=-a^2$

$\Rightarrow m=\pm\, ai$

C.F.: $A\cos ax+B\sin ax$

$$\text{P.I}=\frac{1}{(D+ai)(D-ai)}\sec ax\rightarrow(1)$$

Using partial fractions

$$\frac{1}{D^2+a^2}=\left[\frac{C_1}{D+ai}+\frac{C_2}{D-ai}\right]$$

$$1=C_1(D-ai)+C_2(D+ai)$$

$$C_1=-\frac{1}{2ia},\ C_2=\frac{1}{2ia}$$

$$\text{P.I.} = -\frac{1}{2ia}\frac{1}{(D+ai)}\sec ax + \frac{1}{2ia}\frac{1}{(D-ai)}\sec a\,x$$

$$= -\frac{1}{2ia}\frac{1}{D-(-ai)}\sec ax + \frac{e^{aix}}{2ia}\int e^{-aix}\sec a\,x\,dx$$

$$= -\frac{e^{-aix}}{2ia}\int e^{aix}\sec a\,x\,dx + \frac{e^{aix}}{2ia}\int e^{-aix}\sec a\,x\,dx$$

$$\text{P.I.} = -\frac{e^{-aix}}{2ia}\int \frac{(\cos ax + \mathrm{i}\sin ax)}{\cos ax}dx + \frac{e^{aix}}{2ia}\int \frac{(\cos ax - \mathrm{i}\sin ax)}{\cos ax}dx$$

$$\text{P.I.} = -\frac{e^{-aix}}{2ia}\int (1 + i\tan ax)\,dx + \frac{e^{aix}}{2ia}\int (1 - i\tan ax)\,dx$$

$$= -\frac{e^{-aix}}{2ia}\left[x + \frac{i}{a}\log\sec ax\right] + \frac{e^{aix}}{2ia}\left[x - \frac{i}{a}\log\sec ax\right]$$

$$= \frac{2x}{2a}\left[\frac{e^{aix} - e^{-aix}}{2i}\right] - \frac{2i}{2ia^2}[\log\sec ax]\left[\frac{e^{aix} + e^{-aix}}{2}\right]$$

$$= \frac{x}{a}\sin ax - \frac{1}{a^2}(\log\sec ax)(\cos ax)$$

$$= \frac{1}{a^2}[ax\sin ax + \cos ax\log\cos ax]$$

G.S. is $y = C.F + P.I.$

Second-order linear differential equations

Differential equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=Q(x)$

are called second order linear differential equations.

When $Q(x)=0$ then the equations are referred to as homogeneous, When

$Q(x)\neq 0$ then the equations are non-homogeneous.

Note that the general solution to such an equation must include two arbitrary constants to be completely general.

Theorem

If $y = f(x)$ and $y = g(x)$ are two solutions then so is $y = f(x) + g(x)$

Proof

we have $a\dfrac{d^2 f}{dx^2} + b\dfrac{df}{dx} + cf = 0$ and $a\dfrac{d^2 g}{dx^2} + b\dfrac{dg}{dx} + cg = 0$

Adding: $$a\frac{d^2 f}{dx^2} + b\frac{df}{dx} + cf + a\frac{d^2 g}{dx^2} + b\frac{dg}{dx} + cg = 0$$

$$a\left(\frac{d^2 f}{dx^2} + \frac{d^2 g}{dx^2}\right) + b\left(\frac{df}{dx} + \frac{dg}{dx}\right) + c(f + g) = 0$$

And so $y = f(x) + g(x)$ is a solution to the differential equation.

$y = Ae^{mx}$, for A and m, is a solution to the equation $b\frac{dy}{dx} + cy = 0$

It is reasonable to consider it as a possible solution for

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$y = Ae^{mx} \Rightarrow \frac{dy}{dx} = Ame^{mx} \Rightarrow \frac{d^2y}{dx^2} = Am^2e^{mx}$$

If $y = Ae^{mx}$ is a solution it must satisfy $aAm^2e^{mx} + bAme^{mx} + cAe^{mx} = 0$

assuming $Ae^{mx} \neq 0$, then by division we get $am^2 + bm + c = 0$

The solutions to this quadratic will provide two values of m which will make
y = Ae^{mx} a solution.

If we call these two values m_1 and m_2, then we have two solutions.

$y = Ae^{m_1 x}$ and $y = Be^{m_2 x}$

A and B are used to distinguish the two arbitrary constants.

From the theorem given previously;

$y = Ae^{m_1 x} + Be^{m_2 x}$ Is a solution.

The two arbitrary constants needed for second order differential equations ensure all solutions are covered.

The equation $am^2 + bm + c = 0$ is called the auxiliary equation.

The type of solution we get depends on the nature of the roots of this equation.

When roots are real and distinct

Find the general solution of $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$

The auxiliary equation is

$$m^2 - 5m + 6 = 0$$
$$(m-2)(m-3) = 0$$
$$m = 2, \quad or\ m = 3$$

Thus the general solution is $y = Ae^{2x} + Be^{3x}.$

To find a particular solution we must be given enough information.

Roots are complex conjugates

When the roots of the auxiliary equation are complex, they will be of the form $m_1 = p + iq$ *and* $m_2 = p - iq$. Hence the general equation will be

$$
\begin{aligned}
y &= Ae^{(p+iq)x} + Be^{(p-iq)x} \\
&= Ae^{px}e^{iqx} + Be^{px}e^{-iqx} \\
&= e^{px}\left(Ae^{iqx} + Be^{-iqx}\right)
\end{aligned}
$$

We know that $e^{i\theta} = \cos\theta + i\sin\theta$

$$
\begin{aligned}
&= e^{px}\left(A(\cos qx + i\sin qx) + B\left(\cos(-qx) + i\sin(-qx)\right)\right) \\
&= e^{px}\left(A(\cos qx + i\sin qx) + B\left(\cos qx - i\sin qx\right)\right) \\
&= e^{px}\left((A+B)\cos qx + (A-B)i\sin qx\right) \\
&= e^{px}\left(C\cos qx + D\sin qx\right)
\end{aligned}
$$

Where $C = A + B$ and $D = (A - B)i$

Non homogeneous second order differential equations

Non homogeneous equations take the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=Q(x)$$

Suppose *g*(*x*) is a particular solution to this equation. Then

$$a\frac{d^2g}{dx^2}+b\frac{dg}{dx}+cg=Q(x)$$

Now suppose that *g*(*x*) + *k*(*x*) is another solution. Then

$$a\frac{d^2(g+k)}{dx^2}+b\frac{d(g+k)}{dx}+c(g+k)=Q(x)$$

Giving

$$a\frac{d^2g}{dx^2} + a\frac{d^2k}{dx^2} + b\frac{dg}{dx} + b\frac{dk}{dx} + cg + ck = Q(x)$$

$$\Rightarrow \left(a\frac{d^2g}{dx^2} + b\frac{dg}{dx} + cg\right) + \left(a\frac{d^2k}{dx^2} + b\frac{dk}{dx} + ck\right) = Q(x)$$

$$\Rightarrow Q(x) + \left(a\frac{d^2k}{dx^2} + b\frac{dk}{dx} + ck\right) = Q(x)$$

$$\Rightarrow a\frac{d^2k}{dx^2} + b\frac{dk}{dx} + ck = 0$$

From the work in previous exercises we know how to find *k*(*x*).

This function is referred to as the Complimentary Function.

(CF) The function *g*(*x*) is referred to as the Particular Integral.

(PI)

General Solution = CF + PI

Find the general solution to $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 15x - 7,$

given that the PI is of the form $k(x) = Px + Q$

Finding the (CF): the auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$$\Rightarrow (m-3)(m-2) = 0$$

$$m = 2 \text{ or } m = 3$$

Thus the CF is $y = Ae^{2x} + Be^{3x}$

Finding the PI: $y = Px + Q \Rightarrow \frac{dy}{dx} = P \Rightarrow \frac{d_2y}{dx^2} = 0$

Substituting into the original equation

$$0 - 5P + 6(Px + Q) = 15x - 7$$

$$\Rightarrow 6Px + 6Q - 5P = 15x - 7$$

$$\Rightarrow 6P = 15 \quad \Rightarrow p = \frac{5}{2}$$

$$\Rightarrow Q = \frac{11}{12}$$

Queries …?