Department of Engg. Mathematics

Course : Engineering Mathematics-II (17MAT21).

Sem.: 2nd

Laplace Transform

Content

- Laplace Transform for ODEs
- Table of selected Laplace Transforms
- step functions in Laplace
- Properties of Laplace Transforms
- Properties: Linearity
- Properties: Scaling in Time
- Properties: Time Shift
- Properties: S-plane (frequency) shift
- Properties: Multiplication by t^n
- Real-Life Applications

The French Newton
Pierre-Simon Laplace

- Developed mathematics in astronomy, physics, and statistics
- Began work in calculus which led to the Laplace Transform
- Focused later on celestial mechanics
- One of the first scientists to suggest the existence of black holes

Why use Laplace Transforms?

- Find solution to differential equation using algebra
- Relationship to Fourier Transform allows easy way to characterize systems
- No need for convolution of input and differential equation solution
- Useful with multiple processes in system

How to use Laplace

- Find differential equations that describe system
- Obtain Laplace transform
- Perform algebra to solve for output or variable of interest
- Apply inverse transform to find solution

What are Laplace transforms?

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

1. t is real, s is complex!
2. Note “transform”: f(t) $\rightarrow$ F(s), where t is integrated and s is variable
3. Conversely F(s) $\rightarrow$ f(t), t is variable and s is integrated
4. Assumes f(t) = 0 for all t < 0

Laplace Transform Theory

- General Theory

$$F(s) = L(f(t)) = \int_0^{+\infty} e^{-st} f(t)\,dt = \lim_{\tau \to \infty} \int_0^{\tau} e^{-st} f(t)\,dt$$

- Example

$$f(t) = 1$$

$$L(f(t)) = \int_0^{+\infty} e^{-st}\, 1\,dt = \lim_{\tau \to \infty} \left(\frac{e^{-st}}{-s} \Bigg|_0^{\tau} \right)$$

$$= \lim_{\tau \to \infty} \left(\frac{e^{-s\tau}}{-s} + \frac{1}{s} \right) = \frac{1}{s}$$

Laplace Transform for ODEs

- **Equation with initial conditions**
- **Laplace transform is linear**
- **Apply derivative formula**

$$\frac{d^2y}{dt^2} + y = 1, \qquad y(0) = y'(0) = 0$$

$$L(y'') + L(y) = L(1)$$

$$s^2 L(y) - sy(0) - y'(0) + L(y) = \frac{1}{s}$$

$$L(y) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

Table of selected Laplace Transforms

$$f(t) = u(t) \Leftrightarrow F(s) = \frac{1}{s}$$

$$f(t) = e^{-at}u(t) \Leftrightarrow F(s) = \frac{1}{s+a}$$

$$f(t) = \cos(t)u(t) \Leftrightarrow F(s) = \frac{s}{s^2+1}$$

$$f(t) = \sin(t)u(t) \Leftrightarrow F(s) = \frac{1}{s^2+1}$$

More transforms

$$f(t) = t^n u(t) \Leftrightarrow F(s) = \frac{n!}{s^{n+1}}$$

$$n = 0, f(t) = u(t) \Leftrightarrow F(s) = \frac{0!}{s^1} = \frac{1}{s}$$

$$n = 1, f(t) = tu(t) \Leftrightarrow F(s) = \frac{1!}{s^2}$$

$$n = 5, f(t) = t^5 u(t) \Leftrightarrow F(s) = \frac{5!}{s^6} = \frac{120}{s^6}$$

$$f(t) = \delta(t) \Leftrightarrow F(s) = 1$$

Note on step functions in Laplace

- Unit step function definition:

$$u(t) = 1, \ t \geq 0$$
$$u(t) = 0, \ t < 0$$

- Used in conjunction with f(t) $\rightarrow$ f(t)u(t) because of Laplace integral limits:

Properties of Laplace Transforms

- Linearity
- Scaling in time
- Multiplication by t^n
- Integration
- Differentiation

Properties: Linearity

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 F_1(s) + c_2 F_2(s)$$

Example :

$$L\{\sinh(t)\} =$$

$$y\{\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}\} =$$

$$\frac{1}{2}L\{e^{t}\} - \frac{1}{2}L\{e^{-t}\} =$$

$$\frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) =$$

$$\frac{1}{2}\left(\frac{(s+1)-(s-1)}{s^2-1}\right) = \frac{1}{s^2-1}$$

Proof :

$$L\{c_1 f_1(t) + c_2 f_2(t)\} =$$

$$\int_0^\infty [c_1 f_1(t) + c_2 f_2(t)]e^{-st}dt =$$

$$c_1\int_0^\infty f_1(t)e^{-st}dt + c_2\int_0^\infty f_2(t)e^{-st}dt =$$

$$c_1 F_1(s) + c_2 F_2(s)$$

Properties: Scaling in Time

$$L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Example :

$$L\{\sin(\omega t)\}$$

$$\frac{1}{\omega}\left(\frac{1}{(s/\omega)^2}+1\right) =$$

$$\frac{1}{\omega}\left(\frac{\omega^2}{s^2+\omega^2}\right) =$$

$$\frac{\omega}{s^2+\omega^2}$$

Proof :

$$L\{f(at)\} = \int_0^{\infty} f(at)e^{-st}dt =$$

let $u = at, t = \frac{u}{a}, dt = \frac{1}{a}du$

$$\frac{1}{a}\int_0^{\frac{\infty}{a}} f(u)e^{-(\frac{s}{a})u}du =$$

$$\frac{1}{a}F\left(\frac{s}{a}\right)$$

Properties: Time Shift

$$L\{f(t-t_0)u(t-t_0)\} = e^{-st_0}F(s)$$

Example : $L\{e^{-a(t-10)}u(t-10)\} = \frac{e^{-10s}}{s+a}$

Proof : $L\{f(t-t_0)u(t-t_0)\} =$

$$\int_0^{\infty} f(t-t_0)u(t-t_0)e^{-st}dt =$$

$$\int_{t_0}^{\infty} f(t-t_0)e^{-st}dt =$$

let $u = t - t_0, t = u + t_0$

$$\int_0^{\infty - t_0} f(u)e^{-s(u+t_0)}du =$$

$$e^{-st_0}\int_0^{\infty} f(u)e^{-su}du = e^{-st_0}F(s)$$

Properties: S-plane (frequency) shift

$$L\{e^{-at} f(t)\} = F(s+a)$$

Example :

$$L\{e^{-at} \sin(\omega t)\} = \frac{\omega}{(s+a)^2 + \omega^2}$$

Proof :

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-at} f(t) e^{-st} dt = \int_0^\infty f(t) e^{-(s+a)t} dt = F(s+a)$$

Properties: Multiplication by t^n

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Example :

$$L\{t^n u(t)\} = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right) = \frac{n!}{s^{n+1}}$$

Proof :

$$L\{t^n f(t)\} = \int_0^\infty t^n f(t) e^{-st} dt = \int_0^\infty f(t) t^n e^{-st} dt = (-1)^n \int_0^\infty f(t) \frac{\partial^n}{\partial s^n} e^{-st} dt = (-1)^n \frac{\partial^n}{\partial s^n} \int_0^\infty f(t) e^{-st} dt = (-1)^n \frac{\partial^n}{\partial s^n} F(s)$$

The "D" Operator

1. Differentiation shorthand

2. Integration shorthand

if

then

if

then

Difference in

$$f(0^+), f(0^-) \ \& \ f(0)$$

- The values are only different if f(t) is not continuous @ t=0
- Example of discontinuous function: u(t)

$$f(0^-) = \lim_{t \to 0^-} u(t) = 0$$

$$f(0^+) = \lim_{t \to 0^+} u(t) = 1$$

$$f(0) = u(0) = 1$$

Properties: Nth order derivatives

$$L\{D^2 f(t)\} = ?$$

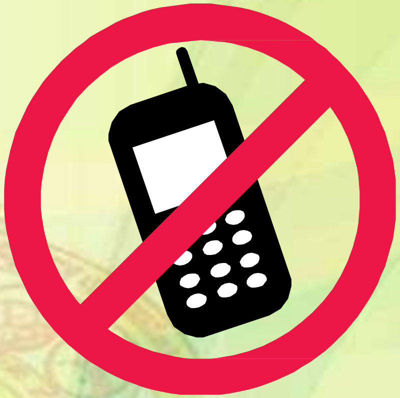
let $$g(t) = Df(t), g(0) = Df(0) = f'(0)$$
$$= L\{D^2 g(t)\} = sG(s) - g(0) = sL\{Df(t)\} - f'(0)$$
$$= s(sF(s) - f(0)) - f'(0) = s^2 F(s) - sF(0) - f'(0)$$

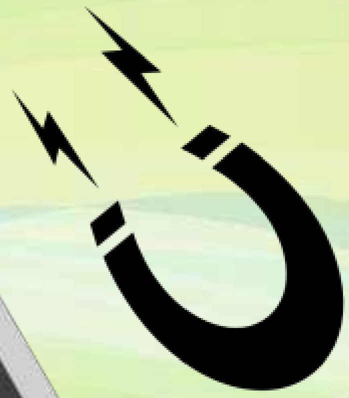
$$L\{D^n f(t)\} = s^n F(s) - s^{(n-1)} f(0) - s^{(n-2)} f'(0) - \cdots - sf^{(n-2)'}(0) - f^{(n-1)'}(0)$$

NOTE: to take $L\{D^n f(t)\}$
you need the value @ t=0 for
$D^{n-1} f(t), D^{n-2} f(t), \ldots Df(t), f(t) \rightarrow$ **called initial conditions!**
We will use this to solve differential equations!

Real-Life Applications

- Semiconductor mobility
- Call completion in wireless networks
- Vehicle vibrations on compressed rails
- Behavior of magnetic and electric fields above the atmosphere





Queries ....?