Department of Engg. Mathematics

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Complex Variable

Content

- Recapitulation of Basic Concepts
- Cauchy-Riemann conditions
- Cauchy's integral theorem
- Conformal Mapping

Cauchy-Riemann conditions

Complex algebra

Complex number: $z = x + iy$ (both x and y are real, $i = \sqrt{-1}$.)

Complex algebra:

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \quad \text{(Anologous to 2d vectors.)}$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \quad (\Rightarrow cz = c(x+iy) = cx + icy) \quad (\Rightarrow z_1 - z_2)$$

$$z^* = (x+iy)^* = x - iy$$

Complex conjugation: $\Rightarrow zz^* = (x+iy)(x-iy) = x^2 + y^2$

Polar representation: $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

Modulus (magnitude): $|z| = \sqrt{zz^*} = r = \sqrt{x^2 + y^2} \qquad \Rightarrow |z_1 z_2| = |z_1||z_2|$

Argument (phase):

$$\arg(z) = \theta = \arctan\left(\frac{y}{x}\right) \quad (+\pi \text{ if } z \text{ is in the 2nd or 3rd quadrants.})$$

$$\Rightarrow \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

Functions of a complex variable:

All elementary functions of real variables may be extended into the complex plane.

$$\text{Example}: \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \rightarrow \quad e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

A complex function can be resolved into its *real part* and *imaginary part*: $f(z) = u(x,y) + iv(x,y)$

$$\text{Examples}: \quad z^2 = (x+iy)^2 = (x^2+y^2) + i2xy$$

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

Multi-valued functions and branch cuts:

$$\text{Example 1}: \ln z = \ln(re^{i\theta}) = \ln[re^{i(\theta+2n\pi)}] = \ln r + i(\theta+2n\pi) = u + iv$$

To remove the ambiguity, we can limit all phases to $(-\pi,\pi)$.

$\theta = -\pi$ is the ***branch cut*****.**

ln$z$ with n = 0 is the ***principle value*****.**

$$\text{Example 2}: z^{1/2} = (re^{i\theta})^{1/2} = \left[re^{i(\theta+2n\pi)}\right]^{1/2} = r^{1/2}e^{i(\theta+2n\pi)/2}$$

We can let z move on 2 **Riemann sheets** so that is single valued everywhere.

Cauchy-Riemann conditions

Analytic functions: If $f(z)$ is differentiable at $z = z_0$ and within the neighborhood of $z=z_0$, $f(z)$ is said to be **analytic** at $z = z_0$. A function that is analytic in the whole complex plane is called an *entire function*.

Cauchy-Riemann conditions for differentiability

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta f(z)}{\Delta z}$$

In order to let f be differentiable, $f'(z)$ must be the same in any direction of Δz.

Particularly , it is necessary that

$$\text{For } \Delta z = \Delta x, \quad f'(z) = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

$$\text{For } \Delta z = i\Delta y, \quad f'(z) = \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Equating them we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

← Cauchy-Riemann conditions

Conversely, if the Cauchy-Riemann conditions are satisfied, *f* (*z*) is differentiable:

$$\frac{df}{dz}=\lim_{\Delta z\to 0}\frac{\Delta f(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\Delta y}{\Delta x+i\Delta y}=\lim_{\Delta z\to 0}\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\Delta y}{\Delta x+i\Delta y}$$

$$=\lim_{\Delta z\to 0}\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\Delta x+i\Delta y)}{\Delta x+i\Delta y}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x},\quad \text{and}=\frac{1}{i}\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right).$$

More about Cauchy-Riemann conditions:

1) It is a very strong restraint to functions of a complex variable.

2) $$\frac{df}{dz}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=\frac{\partial u}{\partial(iy)}+i\frac{\partial v}{\partial(iy)}.$$

3) $$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}=0\Rightarrow\nabla u\cdot\nabla v=0\Rightarrow\nabla u\perp\nabla v\Rightarrow u=c_1\perp v=c_2$$

4) Equivalent to $\frac{\partial f}{\partial z^*}=0$, so that $f(z,z^*)$ only depends on z:

$$\frac{\partial f}{\partial z^*}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*}=\frac{\partial f}{\partial x}\frac{1}{2}+\frac{\partial f}{\partial y}\left(-\frac{1}{2i}\right)=0\Rightarrow\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}=0\Rightarrow\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)=0\Rightarrow\cdots$$

e.g., $f=x-iy$ is everywhere continuous but not analytic.

Cauchy-Riemann conditions:

Our Cauchy-Riemann conditions were derived by requiring *f* '(*z*) be the same when *z* changes along *x* or *y* directions. How about other directions? Here I do a general search for the conditions of differentiability.

$$f'(z)=\frac{df}{dz}=\frac{du+idv}{dx+idy}=\frac{\left(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\right)+i\left(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\right)}{dx+idy}=\frac{\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}\right)+i\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{dy}{dx}\right)}{1+i\frac{dy}{dx}}$$

Now let $\frac{dy}{dx}=p$, the direction of the change of z. We want to find the condition under whic h $f'(z)$ does not depend on p.

$$\frac{df'(z)}{dp}=0=\frac{d}{dp}\frac{\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}p\right)+i\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}p\right)}{1+ip}=\frac{\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)(1+ip)-i\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}p\right)+\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}p\right)}{(1+ip)^2}$$

$$=\frac{\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)}{(1+ip)^2}\Rightarrow\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases}$$

That is, if we require $f'(z)$ be the same at all directions , we get the same Cauchy - Riemann conditions .

Cauchy's theorem

Cauchy's integral theorem

Contour integral:

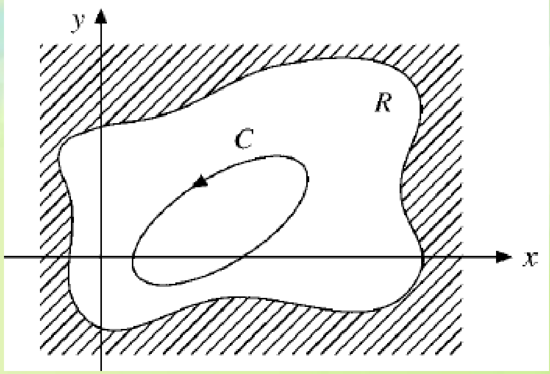
$$\int_{z_1}^{z_2} f(z)dz = \int_C (u+iv)(dx+idy) = \int_C (udx - vdy) + i\int_C (vdx+udy)$$

Cauchy's integral theorem: If *f* (*z*) is analytic in a simply connected region *R*, [and *f* ′(*z*) is continuous throughout this region,] then for any closed path *C* in *R*, the contour

integral of *f* (*z*) around *C* is zero:

Proof using Stokes' theorem $\oint_C \mathbf{V}\cdot d\boldsymbol{\lambda} = \iint_S \nabla\times\mathbf{V}\cdot d\boldsymbol{\sigma}$

$$\oint_C \left(V_x dx + V_y dy\right) = \iint_S \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dxdy$$



$$\oint_C f(z)dz = \oint_C (udx - vdy) + i\oint_C (vdx+udy)$$

$$= \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$

$$= 0$$

Cauchy's integral formula

Cauchy's integral formula:

If $f(z)$ is analytic within and on a closed contour C, then for any point z_0 within C,

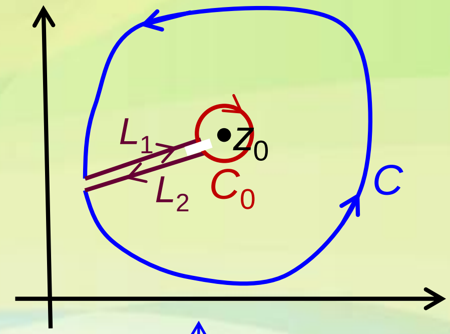
$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz$$

Proof :

$$\oint_C \frac{f(z)}{z-z_0}dz + \oint_{L_1} \frac{f(z)}{z-z_0}dz + \oint_{C_0} \frac{f(z)}{z-z_0}dz + \oint_{L_2} \frac{f(z)}{z-z_0}dz = 0$$

$$\oint_C \frac{f(z)}{z-z_0}dz = -\oint_{C_0} \frac{f(z)}{z-z_0}dz = -\int_{2\pi}^{0} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta} d\theta \quad (\text{Let } r \to 0)$$

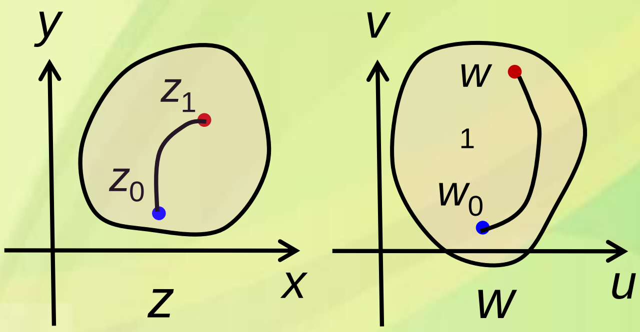
$$= 2\pi i f(z_0)$$



Can directly use the contour deformation theorem.

Mapping

Mapping



Mapping: A complex function
can be thought of as describing a mapping from the complex *z*-plane into the complex *w*-plane.
In general, a point in the *z*-plane is mapped into a point in the *w*-plane. A curve in the *z*-plane is mapped into a curve in the *w*-plane. An area in the *z*-plane is mapped into an area in the *w*-plane.

Examples of mapping:

Translation:

$$w = z + z_0$$

Rotation:

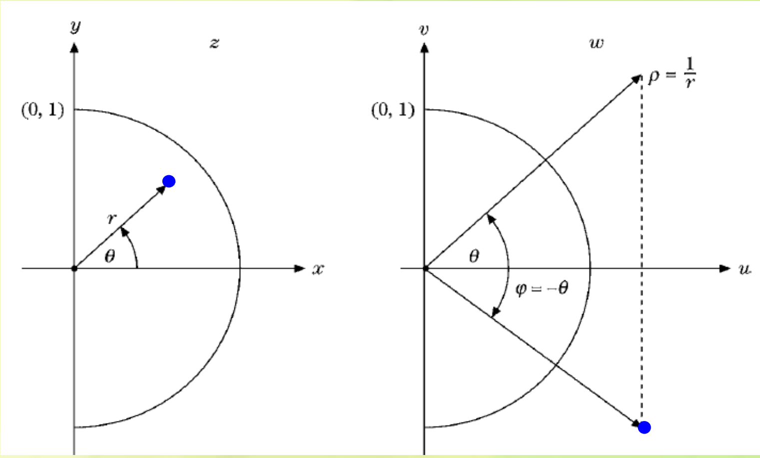
$$w = zz_0, \text{ or}$$

$$\rho e^{i\varphi} = re^{i\theta} \cdot r_0 e^{i\theta_0} \Rightarrow \begin{cases} \rho = r \cdot r_0 \\ \varphi = \theta + \theta_0 \end{cases}$$

Inversion:

$$w = \frac{1}{z}, \text{ or}$$

$$\rho e^{i\varphi} = \frac{1}{re^{i\theta}} \Rightarrow \begin{cases} \rho = \frac{1}{r} \\ \varphi = -\theta \end{cases}$$



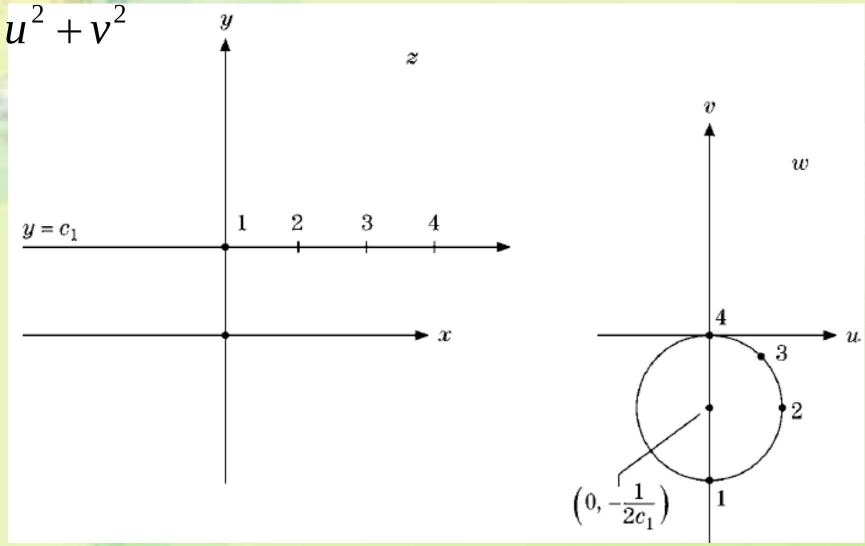
In Cartesian coordinates:

$$w = \frac{1}{z} \Rightarrow u + iv = \frac{1}{x+iy} \Rightarrow \begin{cases} u = \frac{x}{x^2+y^2} \\ v = -\frac{y}{x^2+y^2} \end{cases}, \begin{cases} x = \frac{u}{u^2+v^2} \\ y = -\frac{v}{u^2+v^2} \end{cases}.$$

A straight line is mapped into a circle:

$$y = ax + b \Rightarrow -\frac{v}{u^2+v^2} = \frac{au}{u^2+v^2} + b$$

$$\Rightarrow b(u^2+v^2) + au + v = 0.$$



Conformal mapping

Conformal mapping: The function $w(z)$ is said to be conformal at z_0 if it preserves the angle between any two curves through z_0.

If $w(z)$ is analytic and $w'(z_0) \neq 0$, then $w(z)$ is conformal at z_0.

Proof: Since $w(z)$ is analytic and $w'(z_0) \neq 0$, we can expand $w(z)$ around $z = z_0$ in a Taylor series:

$$w = w(z_0) + w'(z_0)(z - z_0) + \frac{1}{2} w''(z_0)(z - z_0)^2 + \cdots$$

$$\lim_{z - z_0 \to 0} \frac{w - w_0}{z - z_0} = w'(z_0), \text{ or } w - w_0 \approx w'(z_0)(z - z_0).$$

$$w - w_0 = Ae^{i\alpha}(z - z_0) \Rightarrow \varphi = \alpha + \theta \Rightarrow \varphi_2 - \varphi_1 = \theta_2 - \theta_1.$$

1) At any point where $w(z)$ is conformal, the mapping consists of a rotation and a dilation.
2) The local amount of rotation and dilation varies from point to point. Therefore a straight line is usually mapped into a curve.
3) A curvilinear orthogonal coordinate system is mapped to another curvilinear orthogonal coordinate system .

What happens if $w'(z_0) = 0$?

Suppose $w^{(n)}(z_0)$ is the first non-vanishing derivative at z_0.

$$w - w_0 \approx \frac{w^{(n)}(z_0)}{n!}(z - z_0)^n \Rightarrow \rho e^{i\varphi} = \frac{1}{n!} B e^{i\beta}(r e^{i\theta})^n \Rightarrow \begin{cases} \rho = \dfrac{Br^n}{n!} \\ \varphi = n\theta + \beta \end{cases}$$

This means that at $z = z_0$ the angle between any two curves is magnified by a factor n and then rotated by β.

Queries ….?