Department of Engg. Mathematics

Course : Engg. Mathematics-IV 15MAT41. Sem.: 4th (2017-18)

Course Coordinator:

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Numerical Methods and Special Function

Content

- Numerical solution of second order differential equations by

  1) Runge Kutta method of 4^{th} order

  2)Milne's Method.
- Bessel's Functions
- Orthogonality of Bessel functions
- Rodrigue's Formula
- Series solution of Legendre's differential equation leading to $P_n(x)$

Numerical solution of Second order Ordinary differential equations

Consider a Second order O.D.E of the form

$Y'' = F(x, y, y')$ - - - - - (1)

with initial conditions $y(x_0) = y_0$ and $y'(x_0) = y'_0$

Set $y' = z$ and $y'' = z'$

$$\frac{dy}{dx} = z \ and \ \frac{dz}{dx} = f(x, y, z) \ \ with \ y(x_0) = y_0 \ and \ z(x_0) = y_0'$$

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$z_1 = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$

where

$$k_1 = hf[x_0, y_0, z_0]$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$l_1 = hg[x_0, y_0, z_0]$$

$$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$$

Milne's Predictor-Corrector Method

$$y'' = F(x, y, y')$$

with initial conditions $y(x_0) = y_0 \; and \; y'(x_0) = y_0'$

$$y' = z \;\; and \; y'' = z'$$

$$\frac{dy}{dx} = z \;\; and \;\; \frac{dz}{dx} = g(x, y, z) \;\; with \;\; y(x_0) = y_0 \; and \; z(x_0) = y_0'$$

$$y_p = y_0 + \frac{4h}{3}(2z_1 - z_2 + z_3)$$

$$z_p = z_0 + \frac{4h}{3}(2z_1' - z_2' + z_3')$$

$$y_c = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_c = z_2 + \frac{4h}{3}(z_2' + z_3' + z_4')$$

BESSEL FUNCTIONS

Definition: The second order differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

Where n is a constant, is called Bessel's equation of order n. It is one of the most important differential equations in applied mathematics. Its particular solutions are Bessel functions.

$$J_n(x) = \sum_0^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} \quad when\ n\ is\ positive$$

$$J_{-n}(x) = \sum_0^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{n+2r} \quad when\ n\ is\ negative$$

$$y = AJ_n(x) + BJ_{-n}(x)$$

where A and B are arbitrary constants

Orthogonality of Bessel functions

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = f(x) = \begin{cases} 0, & \alpha \neq \beta \\ 1/2[J_{n+1}(x)]^2, & \alpha = \beta \end{cases} \quad [J_{n+1}(x)]^2$$

Where α, β are the roots of $J_n(x) = 0$.

Rodrigue's Formula

The relation,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

is known as Rodrigue's formula

Series solution of Legendre's differential equation leading to

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}[3x^2 - 1]$$

$$P_3(x) = \frac{1}{2}[5x^3 - 3x]$$

$$P_4(x) = -\frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$$

Queries ….?