Department of Engg. Mathematics

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Numerical Methods

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Numerical Methods:

Algorithms that are used to obtain numerical solutions of a mathematical problem.

Why do we need them?

1. No analytical solution exists,
2. An analytical solution is difficult to obtain or not practical.

Where are Numerical Methods Used?

- **Thermo and Heat Transfer: compute the flow of heat (diffusion) through a media (Finite Element Methods)**
- **Fluid Dynamics: solve the Navier-Stokes equations (Finite Element/Finite Volume Methods)**
- **Mechanics of Materials: solve the partial differential equations to find stress/strain distribution (Finite Element Methods)**
- **Multibody Dynamics: solve differential-algebraic equations (DAEs) that result from Newton's second law**

Contents

- Taylor’s series method
- Modified Euler’s method
- Runge - Kutta method of fourth order
- Milne’s and Adams-Bashforth predictor and corrector methods

Taylor Series

The Taylor series expansion of $f(x)$ about a:

$$f(a) + f^{'}(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots$$

or

$$Taylor\ Series = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)\,(x-a)^k$$

If the series converge, we can write:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(a)\,(x-a)^k$$

Example-1:

Obtain Taylor series expansion of $f(x) = \frac{1}{x}$ at $a = 1$

$$f(x) = \frac{1}{x} \qquad f(1) = 1$$

$$f'(x) = \frac{-1}{x^2} \qquad f'(1) = -1$$

$$f^{(2)}(x) = \frac{2}{x^3} \qquad f^{(2)}(1) = 2$$

$$f^{(3)}(x) = \frac{-6}{x^4} \qquad f^{(3)}(1) = -6$$

Taylor Series Expansion $(a = 1)$: $1 - (x-1) + (x-1)^2 - (x-1)^3 + \ldots$

Example-2:

Obtain Taylor series expansion of $f(x) = e^{2x+1}$, $a = 0.5$

$$f(x) = e^{2x+1} \qquad f(0.5) = e^{2}$$

$$f'(x) = 2e^{2x+1} \qquad f'(0.5) = 2e^{2}$$

$$f^{(2)}(x) = 4e^{2x+1} \qquad f^{(2)}(0.5) = 4e^{2}$$

$$f^{(k)}(x) = 2^{k}\ e^{2x+1} \qquad f^{(k)}(0.5) = 2^{k} e^{2}$$

$$e^{2x+1} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0.5)}{k!} (x-0.5)^{k}$$

$$= e^{2} + 2e^{2}(x-0.5) + 4e^{2} \frac{(x-0.5)^{2}}{2!} + \ldots + 2^{k} e^{2} \frac{(x-0.5)^{k}}{k!} + \ldots$$

Taylor Series in Two Variables

Let y **= f(x) be a solution of the equation**

$$\frac{dy}{dx} = f(x, y) \quad y(x_o) = y_o$$

Expanding it by Taylor's series about $x = x_0$ **we get**

$$Y = f(x) = Y_o + \frac{(X - X_0)}{1!}Y_1 + \frac{(X - X_0)^2}{2!}Y_2 + \frac{(X - X_0)^3}{3!}Y_3$$

4^{th} Order Runge-Kutta

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{h}{2},\quad y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + \frac{h}{2},\quad y_i + \frac{1}{2}k_2 h)$$

$$k_4 = f(x_i + h,\quad y_i + k_3 h)$$

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Local error is $O(h^5)$ and global error is $O(h^4)$

Example
4^{th}-Order Runge-Kutta Method

$$\frac{dy}{dx} = 1 + y + x^2$$

$$y(0) = 0.5$$

$$h = 0.2$$

$$Use\ RK4\ to\ compute\ \ y(0.2)\ and\ \ y(0.4)$$

4^{th} Order Runge-Kutta

Problem :

$$\frac{dy}{dx} = 1 + y + x^2, \quad y(0) = 0.5$$

$$Use\ RK4\ to\ find\ \ y(0.2), y(0.4)$$

Problem :

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

Use RK 4 to find $y(0.2), y(0.4)$

$$h = 0.2$$

$$f(x, y) = 1 + y + x^2$$

$$x_0 = 0, \quad y_0 = 0.5$$

$$k_1 = f(x_0, y_0) = (1 + y_0 + x_0{}^2) = 1.5$$

$$k_2 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1 h) = 1 + (y_0 + 0.15) + (x_0 + 0.1)^2 = 1.64$$

$$k_3 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2 h) = 1 + (y_0 + 0.164) + (x_0 + 0.1)^2 = 1.654$$

$$k_4 = f(x_0 + h, y_0 + k_3 h) = 1 + (y_0 + 0.16545) + (x_0 + 0.2)^2 = 1.7908$$

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.8293$$

Modified Euler's method

- Consider first order differential equation

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0$$

- Modified Euler's formula is given by

$$y_{(n+1)}^{(r+1)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_{(n+1)}, y_{(n+1)}^{(r)}\right)\right]$$

where $r = 0,1,2,3,4 \ldots.$

- Euler method

  $y_{n+1} = y_n + hf(x_n, y_n)$ Where $x_n = x_0 + nh$

- Euler's Modified Method is

  $y_{n+1} = y_n + h/2\{f(x_n, y_n) + f(x_{n+1}, y_{n+1})\}$

Fourth Order Adams-Bashforth Formula

- More accurate Adams formulas can be obtained by using a higher degree polynomial $P_k(t)$ and more data points.
- For example, the coefficients of a 3rd degree polynomial $P_3(t)$ are found using (t_n, y_n), (t_{n-1}, y_{n-1}), (t_{n-2}, y_{n-2}), (t_{n-3}, y_{n-3}).
- As before, $P_3(t)$ then replaces $\phi'(t)$ in the integral equation

to obtain the fourth order Adams-Bashforth formula

- The local truncation error of this method is proportional to h^5.

Predictor-Corrector Method

- Consider the fourth order Adams-Bashforth and Adams-Moulton formulas, respectively:

$$y_{n+1} = y_n + (h/24)\left(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right)$$
$$y_{n+1} = y_n + (h/24)\left(9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}\right)$$

- Once y_{n-3}, y_{n-2}, y_{n-1}, y_n are known, we compute f_{n-3}, f_{n-2}, f_{n-1}, f_n and use Adams-Bashforth formula (predictor) to obtain y_{n+1}.
- We then compute f_{n+1}, and use the Adams-Bashforth formula (corrector) to obtain an improved value of y_{n+1}.
- We can continue to use corrector formula if the change in y_{n+1} is too large. However, if it is necessary to use the corrector formula more than once or perhaps twice, the step size h is likely too large and should be reduced.

Queries …?