Department of Engg. Mathematics

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Fourier Series

Content

- Periodic Functions
- Fourier Series
- Complex Form of the Fourier Series
- Half-Range Expansion
- Applications

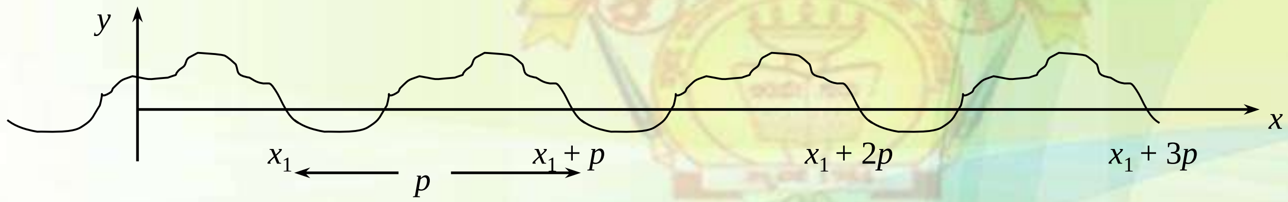
Introduction

Fourier Series ?

A Fourier series is a representation of a function as a series of constants times sine and/or cosine functions of different frequencies.

Periodic Functions

- A function f(x) is said to be periodic if its function values repeat at regular intervals of the independent variables.
- For the following example, a function f(x) has the period p.



In general, a function f(x) is called periodic if there is some positive number p such that ;

$$f(x) = f(x + np)$$

for any integer n.This number p is called a period of f(x).

Fourier Series of a Function

- If f(x) is defined within the interval *c < x < c+2L.* The Fourier Series corresponding to f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

where

$$a_0 = \frac{1}{L}\int_{c}^{c+2L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$b_n = \frac{1}{L}\int_{c}^{c+2L} f(x)\sin\frac{n\pi}{L}x\,dx$$

Dirichlet Conditions

If a function $f(x)$ defined within the interval $c < x < c+2L$, the following conditions must be satisfied;

1. $f(x)$ is defined and single-valued.
2. $f(x)$ is continuous or finite discontinuity in the corresponding periodic interval.
3. $f(x)$ and $f'(x)$ are piecewise continuous .

Odd and Even Functions

- A function f(x) is said to be even if :

$$f(-x) = f(x)$$

i.e the function value for a particular negative value of x is the same as that for the corresponding positive value of x.

- A function f(x) is said to be odd if :

$$f(-x) = -f(x)$$

i.e the function value for a particular negative value of x is numerically equal to that for the corresponding positive value of x but opposite in sign.

If f(x) is an even function

$$a_0 = \frac{1}{a}\int_{-a}^{a} f(x)dx = \frac{2}{a}\int_{0}^{a} f(x)dx$$

$$a_n = \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{n\pi}{a}x\,dx = \frac{2}{a}\int_{0}^{a} f(x)\cos\frac{n\pi}{a}x\,dx$$

$$b_n = \frac{1}{a}\int_{-a}^{a} f(x)\sin\frac{n\pi}{a}x\,dx = 0.$$

$\Rightarrow$ Fourier series, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{a}x\right)$

is called Fourier Cosines series.

If f(x) is an odd function

$$a_0 = \frac{1}{a}\int_{-a}^{a} f(x)\,dx = 0$$

$$a_n = \frac{1}{a}\int_{-a}^{a} f(x)\cos\frac{n\pi}{a}x\,dx = 0$$

$$b_n = \frac{1}{a}\int_{-a}^{a} f(x)\sin\frac{n\pi}{a}x\,dx = \frac{2}{a}\int_{0}^{a} f(x)\sin\frac{n\pi}{a}x\,dx.$$

$\Rightarrow$ Fourier series, $f(x) = \sum_{n=1}^{\infty}\left(b_n \sin\frac{n\pi}{a}x\right)$

is called Fourier Sine series.

Fourier Series

$f(\theta)$ **be a periodic function with period** 2π

The function can be represented by a trigonometric series as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)\theta \, d\theta + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)\theta \, d\theta$$

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = 0$$

$n \neq m$

$$\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = \pi$$

$n = m$

Even and Odd Functions

Even Functions

$$f(-\theta) = f(\theta)$$

Odd Functions

$$f(-\theta) = -f(\theta)$$

Harmonics

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

Some examples of Fourier series…

1. $$\sum_{n=1}^{\infty} \frac{1}{n} \sin nx = \begin{cases} -\tfrac{1}{2}(\pi + x), & -\pi \le x < 0 \\ \tfrac{1}{2}(\pi - x), & 0 \le x < \pi \end{cases}$$

2. $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin nx = \tfrac{1}{2} x, \quad -\pi < x < \pi$$

3. $$\sum_{n=0}^{\infty} \frac{1}{2n+1} \sin(2n+1)x = \begin{cases} -\pi/4, & -\pi < x < 0 \\ \pi/4, & 0 < x < \pi \end{cases}$$

4. $$\sum_{n=1}^{\infty} \frac{1}{n} \cos nx = -\ln\left[2\sin\left(\frac{|x|}{2}\right)\right], \quad -\pi < x < \pi$$

5. $$\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n} \cos nx = -\ln\left[2\cos\left(\frac{x}{2}\right)\right], \quad -\pi < x < \pi$$

6. $$\sum_{n=0}^{\infty} \frac{1}{2n+1} \cos(2n+1)x = \frac{1}{2}\ln\left[\cot\left(\frac{|x|}{2}\right)\right], \quad -\pi < x < \pi$$

Fourier Cosine and Sine Series

Odd function

$$f(-x) = -f(x)$$

$$\sin x$$

$$\int_{-p}^{p} f(x)dx = 0$$

Properties of Even/Odd Functions

(a) The product of two even functions is even.
(b) The product of two odd functions is even.
(c) The product of an even function and an odd function is odd.
(d) The sum (difference) of two even functions is even.
(e) The sum (difference) of two odd functions is odd.

Even function

$$f(-x) = f(x)$$

$$\cos x$$

$$\int_{-p}^{p} f(x)dx = 2\int_{0}^{p} f(x)dx$$

Case1: Odd

If f(x) is an odd function on the interval (-p, p)

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx = 0$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\cos\tfrac{n\pi}{p}x}_{odd}\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\sin\tfrac{n\pi}{p}x}_{even}\,dx = \frac{2}{p}\int_{0}^{p} \underbrace{f(x)\sin\tfrac{n\pi}{p}x}_{even}\,dx$$

Case2: Even

If f(x) is an even function on the interval (-p, p)

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\sin\tfrac{n\pi}{p}x}_{odd}\,dx$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx = \frac{2}{p}\int_{0}^{p} f(x)dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\cos\tfrac{n\pi}{p}x}_{even}\,dx = \frac{2}{p}\int_{0}^{p} \underbrace{f(x)\cos\tfrac{n\pi}{p}x}_{even}\,dx$$

Fourier Cosine and Sine Series

Fourier cosine Series

The Fourier series of an **even** function on the interval (-p, p) is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$

$$a_0 = \frac{2}{p}\int_0^p f(x)dx$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

Fourier sine Series

The Fourier series of an **odd** function on the interval (-p, p) is the **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$

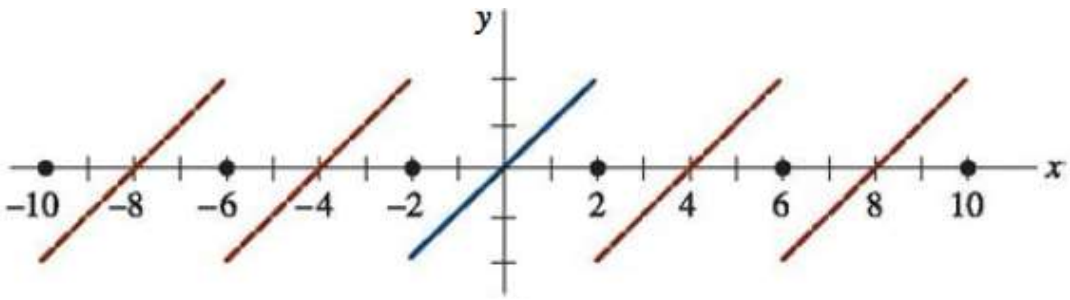
Example

Expand f(x) = x, -2 < x< 2, in a Fourier series.

$$b_n = \frac{4(-1)^{n+1}}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

the series converges to the function on (-2, 2) and the periodic extension (of period 4)



Fourier Cosine and Sine Series

Fourier sine Series

The Fourier series of an **odd** function on the interval (-p, p) is the **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x\,dx$$

Expand in a Fourier series

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 \le x < \pi \end{cases}$$

is odd on the interval.

$$b_n = \frac{2}{\pi} \frac{1-(-1)^n}{n}$$

Fourier Series Applications

- Signal Processing
- Image processing
- Heat distribution mapping
- Wave simplification □
- Light Simplifcation(Interference ,Deffraction etc.)
- Radiation measurements etc.

Queries ...?