Department of Engg. Mathematics

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Fourier Transform and Z-Transform

Content

- Mathematical Background Complex Numbers
- Applications
- Definition
- Properties
- Shifting property
- Fourier Sine and Cosine Transform
- Inverse Fourier Sine and Cosine Transform
- Z-transform of standard functions
- Damping rule

Continued

- **Initial Value Theorem**
- **Inverse Z-Transforms**
- **Standard Inverse Z-Transforms**

Mathematical Background: Complex Numbers

- **A complex number x is of the form:**

$$x = a + jb, \text{ where } j = \sqrt{-1}$$

α: **real part**, b: **imaginary** part

- **Euler's formula:**

$$e^{\pm j\theta} = cos(\theta) \pm jsin(\theta)$$

- Properties

Applications of Fourier Transforms

- X-ray diffraction
- Electron microscopy (and diffraction)
- NMR spectroscopy
- IR spectroscopy
- Fluorescence spectroscopy
- Image processing
- Signal analysis
- Sound filtering
- Partial differential equations

- Fourier transforms have many scientific applications — in physics, number theory, combinatorics, signal processing, probability theory, statistics, cryptography, acoustics, oceanography, optics, geometry, and other areas

Why is FT Useful?

- **Easier** to remove undesirable frequencies in the **frequency** domain.

- **Faster** to perform certain operations in the **frequency** domain than in the **spatial** domain.

What is a Fourier transform?

- **The infinite Fourier transform of Fourier transform of a real valued function f(x) is defined by**

$$F[f(x)] = F(u) = \int_{-\infty}^{\infty} f(x)e^{iux}\ dx$$

- **Inverse Fourier Transform:**

The inverse Fourier Transform of F(u) denoted by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{-iux}\ du$$

Inverse Fourier Sine and Cosine Transform:

- **Inverse Fourier Sine Transform**

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(u)\sin ux \quad du$$

- **Inverse Fourier Cosine Transform**

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(u)\cos ux \quad du$$

- **Property of transforms:**
  - They convert a function from one domain to another with no loss of information
- **Fourier Transform:**

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

converts a function from the time (or spatial) domain to the frequency domain

Properties

$$|e^{\pm j\theta}| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

$$\phi(e^{\pm j\theta}) = \tan^{-1}(\pm \frac{sin(\theta)}{cos(\theta)}) = \tan^{-1}(\pm tan(\theta)) = \pm\theta$$

$$sin(\theta) = \frac{1}{2i}(e^{j\theta} - e^{-j\theta})$$

$$cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

Shifting property

$$F[f(x)] = \hat{f}(u) \; then \; F[f(x-a)] = e^{iua}\hat{f}(u)$$

Fourier Sine and Cosine Transform

$$F_s(u) = \int_0^\infty f(x)\sin ux\,dx$$

$$F_c(u) = \int_0^\infty f(x)\cos ux\,dx$$

Z-transforms

The **z-transform** is the most general concept for the transformation of discrete-time series.

Definition:

If $U_n = f(n)$ defined for all $n = 0, 1, 2, 3 \ldots\ldots\ldots$ and $U_n = 0$ for $n < 0$ then the z-transform of U_n denoted by $Z_T(U_n)$ is defined by

$$Z_T(U_n) = \sum_{n=0}^{\infty} U_n Z^{-n}$$

Z-transform of standard functions

$$Z_T(k^n) = \frac{z}{z-k}$$

$$Z_T(n^2) = \frac{z^2+z}{(z-1)^3}$$

$$Z_T(1) = \frac{z}{z-1}$$

$$Z_T(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}$$

$$Z_T(n) = \frac{z}{(z-1)^2}$$

$$Z_T(k^n n) = \frac{kz}{(z-k)^2}$$

$$Z_T(k^n n^2) = \frac{kz^2 + k^2 z}{(z-k)^3}$$

$$Z_T(k^n n^3) = \frac{kz^3 + 4k^2 z^2 + k^3 z}{(z-k)^4}$$

- **Damping rule (property)**

$$If\ Z_T(u_n) = \bar{u}(z)\ then\quad i)\ Z_T(k^n u_n) = \bar{u}(z/k)\quad ;\ ii)\ Z_T(k^{-n} u_n) = \bar{u}(kz)$$

Shifting Rule

$$Z_T(u_{n+1}) = z[\bar{u}(z) - u_0]$$

$$Z_T(u_{n+2}) = z^2[\bar{u}(z) - u_0 - u_1 z^{-1}]$$

$$Z_T(u_{n+3}) = z^3[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}] \ etc$$

Initial Value Theorem

Statement: If $Z_T(u_n) = \bar{u}(z)$ *then* $\lim_{z \to \infty} \bar{u}(z) = u_0$

Also

$$\lim_{z \to \infty} z[\bar{u}(z) - u_0] = u_1$$

$$\lim_{z \to \infty} z^2[\bar{u}(z) - u_0 - \frac{u_1}{z}] = u_2$$

etc….

Inverse Z-Transforms

$If\ Z_T(u_n) = \bar{u}(z)\ then\ \ Z^{-1}{}_T(\bar{u}(z)) = u_n$

is called inverse Z-transform of

Standard Inverse Z-Transforms

$$z^{-1}{}_T\left[\frac{z}{z-1}\right] = 1$$

$$z^{-1}{}_T\left[\frac{z}{(z-1)^2}\right] = n$$

$$z^{-1}{}_T\left[\frac{z^2+z}{(z-1)^3}\right] = n^2$$

$$z^{-1}{}_T\left[\frac{z^3+4z^2+z}{(z-1)^4}\right] = n^3$$

$$z^{-1}{}_T\left[\frac{z}{z-k}\right] = k^n$$

$$z^{-1}{}_T\left[\frac{kz}{(z-k)^2}\right] = k^n\, n$$

$$z^{-1}{}_T\left[\frac{kz^2+k^2z}{(z-k)^3}\right] = k^n.\, n^2$$

$$z^{-1}{}_T\left[\frac{kz^3+4k^2z^2+k^3z}{(z-k)^4}\right] = k^n.\, n^3$$

Queries …?