Department of Engg. Mathematics

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VECTOR INTEGRATION

Content:

- Basic Definition of Line Integral, Surface Integral & Volume
  Integral.
- Green's Theorem
- Stokes Theorem
- Gauss Divergence Theorem
- Euler's Theorem

Line Integral, Surface & Volume Integral

- **Line Integral** : If there exists a scalar field V along a curve C, then the line integral of V along C is defined by $\int_C V \, d\underset{\sim}{r}$

  where $d\underset{\sim}{r} = dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j} + dz\,\underset{\sim}{k}.$

- **Surface Integral:** If scalar field V exists on surface S, surface integral V of S is defined by

$$\int_S V d\underset{\sim}{S} = \int_S V \underset{\sim}{n}\, dS$$

where,

$$\underset{\sim}{n} = \frac{\nabla S}{|\nabla S|}$$

Volume Integral : If V is a closed region and F is a scalar field in region V, volume integral F of V is

$$\int_V F\,dV = \iiint_V F\,dx\,dy\,dz$$

Example : Scalar function $F = 2x$ defeated in one cubic that has been built by planes $x = 0$, $x = 1$, $y = 0$, $y = 3$, $z = 0$ and $z = 2$. Evaluate volume integral F of the cubic.

- Solution:

$$\int_V F dV = \int_{z=0}^{2} \int_{y=0}^{3} \int_{x=0}^{1} 2x dx dy dz$$

$$= 2\int_{z=0}^{2} \int_{y=0}^{3} \left[\frac{x^2}{2} \right]_0^1 dy dz$$

$$= 2\int_{z=0}^{2} \int_{y=0}^{3} \frac{1}{2} dy dz$$

$$= 2.\frac{1}{2} \int_{z=0}^{2} [y]_0^3 dz$$

$$= \int_{z=0}^{2} 3dz = 3[z]_0^2 = 6$$

Example : Calculate $\int_c \underset{\sim}{F} \cdot d\underset{\sim}{r}$ from A = (0,0,0) to B = (4,2,1) along the curve $x = 4t, y = 2t^2, z = t^3$ if

$$\underset{\sim}{F} = x^2 y\,\underset{\sim}{i} + xz\,\underset{\sim}{j} - 2yz\,\underset{\sim}{k}.$$

⇨ Given $\underset{\sim}{F} = x^2 y\,\underset{\sim}{i} + xz\,\underset{\sim}{j} - 2yz\,\underset{\sim}{k}$

$$= (4t)^2(2t^2)\,\underset{\sim}{i} + (4t)(t^3)\,\underset{\sim}{j} - 2(2t^2)(t^3)\,\underset{\sim}{k}$$

$$= 32t^4\,\underset{\sim}{i} + 4t^4\,\underset{\sim}{j} - 4t^5\,\underset{\sim}{k}.$$

And $d\underset{\sim}{r} = dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j} + dz\,\underset{\sim}{k}$

$$= 4\,dt\,\underset{\sim}{i} + 4t\,dt\,\underset{\sim}{j} + 3t^2 dt\,\underset{\sim}{k}.$$

Then

$$\underset{\sim}{F}.d\underset{\sim}{r} = (32t^4\underset{\sim}{i} + 4t^4\underset{\sim}{j} - 4t^5\underset{\sim}{k})(4\,dt\,\underset{\sim}{i} + 4t\,dt\,\underset{\sim}{j} + 3t^2dt\,\underset{\sim}{k})$$

$$= (32t^4)(4dt) + (4t^4)(4tdt) + (-4t^5)(3t^2dt)$$

$$= 128t^4dt + 16t^5dt - 12t^7dt$$

$$= (128t^4 + 16t^5 - 12t^7)\,dt.$$

At A = (0,0,0), $4t = 0,\ 2t^2 = 0,\ t^3 = 0,$

$$\Rightarrow t = 0.$$

and, at B = (4,2,1), $4t = 4,\ 2t^2 = 2,\ t^3 = 1,$

$$\Rightarrow t = 1.$$

$$\therefore \int_A^B \underset{\sim}{F} \cdot d\,\underset{\sim}{r} = \int_{t=0}^{t=1} (128t^4 + 16t^5 - 12t^7)\,dt$$

$$= \left[\frac{128}{5} t^5 + \frac{8}{3} t^6 - \frac{3}{2} t^8 \right]_0^1$$

$$= \frac{128}{5} + \frac{8}{3} - \frac{3}{2}$$

$$= 26 \frac{23}{30}.$$

GREEN'S THEOREM

Statement : Let *C* be a positively oriented, piecewise-smooth, simple closed curve in the plane
and let *D* be the region bounded by *C*.

If *P* and *Q* have continuous partial derivatives on an open region that contains *D*, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

STOKES' THEOREM

Statement: Let *S* be an oriented piecewise-smooth surface bounded by a simple, closed, piecewise-smooth boundary curve *C* with positive orientation.

Let F be a vector field whose components have continuous partial derivatives on an open region in that contains *S*.

Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

DIVERGENCE THEOREM

Statement :Let *E* be a simple solid region and let *S* be
the boundary surface of *E*, given with +ve orientation.

Let F be a vector field whose component functions have continuous partial derivatives on an open region that contains *E*.

Then,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

STOKE'S VS. GREEN'S THEOREM

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem.

- Green's Theorem relates a double integral over a plane region *D* to a line integral around its plane boundary curve.

- Stokes' Theorem relates a surface integral over a surface *S* to a line integral around the boundary curve of *S* (a space curve).

Euler Equation

- Let's go over what we have shown. We can find a minimum (more generally a stationary point) for the path S if we can find a function for the path that satisfies

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.$$

- The procedure for using this is to set up the problem so that the quantity whose stationary path you seek is expressed as

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx,$$

where $f[y(x), y'(x), x]$ is the function appropriate to your problem. Write down the Euler-Lagrange equation, and solve for the function $y(x)$ that defines the required stationary path.

- Let's do a few examples to make the procedure clear.

The Shortest Path Between Two Points

- We earlier showed that the problem of the shortest path between two points can be expressed as

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\, dx.$$

- The integrand contains our function
- The two partial derivatives in the Euler-Lagrange equation are:

$$\frac{\partial f}{\partial y} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}.$$

- Thus, the Euler-Lagrange equation gives us

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}} = 0.$$

- This says that

$$\frac{y'}{\sqrt{1 + y'^2}} = C, \quad \text{or} \quad y'^2 = C^2(1 + y'^2).$$

- A little rearrangement gives the final result: $y'^2 = \text{constant}$ (call it m^2), so $y(x) = mx + b$. In other words, a straight line is the shortest path.

Queries …?