Department of Engg. Mathematics

Course : Calculus and Linear Algebra (18MAT11). **Sem.: 1st (2018-19)**

Course Coordinator:

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Differential Calculus-1

Content

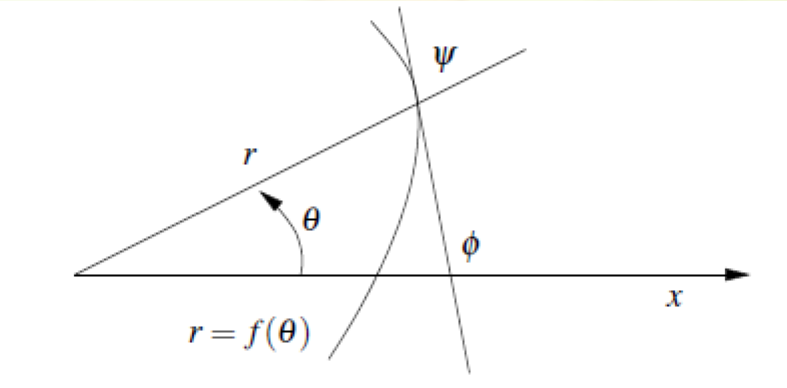
- Polar curves
- Angle between radius vector and tangent
- Angle between two curves
- Pedal equation
- Curvature and Radius of curvature
- Centre and circle of curvature
- The Evolutes of an ellipse

Polar curves

Polar curves are defined by points that are a variable distance from the origin (the pole) depending on the angle measured off the positive x-axis.

Angle between radius vector and tangent

- Consider the angle ψ between the radius vector and the tangent line to a curve, r = f(θ), given in polar coordinates, as shown in Fig



- Figure : The tangent line to the curve r = f(θ) makes an angle of ψ with respect to the radial line at the point of tangency, and an angle φ with respect to the x-axis.

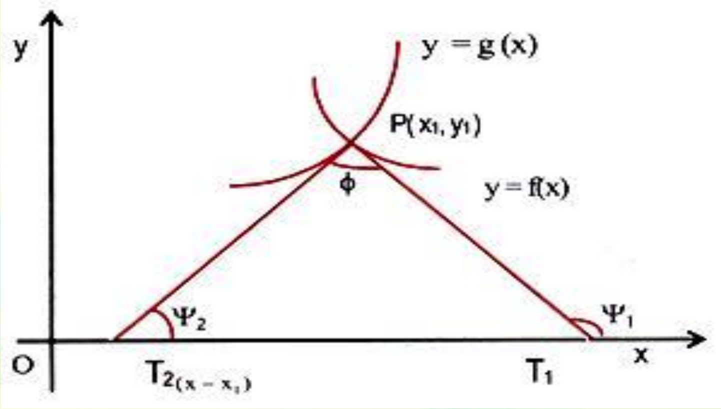
Consider $\phi = \theta + \psi$. Then $r = f(\theta)$ is given in polar coordinates by

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

$$\tan\Psi = \frac{r}{dr/d\theta}$$

Angle between two curves

Let there be two curves $y = f_1(x)$ and $y = f_2(x)$ which intersect each other at point (x_1, y_1). If we draw tangents to these curves at the intersecting point, the angle between these tangents, is called the angle between two curves.



Pedal Equation

The relation between **p** and **r** for a given curve, where **p** is the length of the perpendicular from the origin (or pole) to the tangent at any point on the curve and **r** is the distance of the point from the origin (or pole), is called the pedal equation of the curve.

We know $$\frac{1}{p^2} = \frac{1}{r^2} = + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \qquad (5)$$

Polar Curves

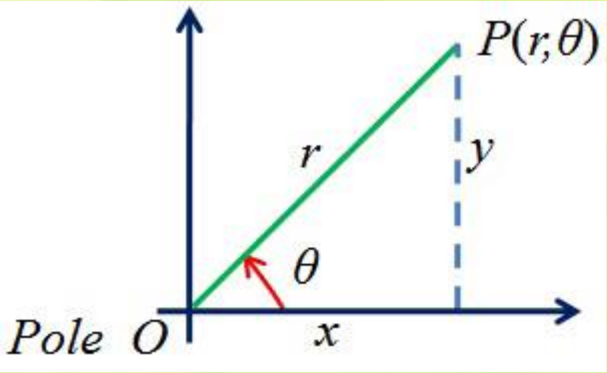
If we traverse in a hill section where the road is not straight, we often see caution boards hairpin bend ahead, sharp bend ahead etc. This gives an indication of the difference in the amount of bending of a road at various points which is the curvature at various points. In this chapter we discuss about the curvature, radius of curvature etc.

Consider a point P in the *xy-Plane*.

r = length of OP= radial distance

θ = Polar angle

(*r, θ)→ Polar co-ordinates*



Let r = f (θ) be the polar curve

$$r = \sqrt{x^2 + y^2}, \qquad \theta = tan^{-1}\left(\frac{y}{x}\right) \_\_\_\_\_\_\_\_\_\_\_\_ (1)$$

$x = r\,Cos\,\theta \;\; ; \;\; y = r\,Sin\,\theta$

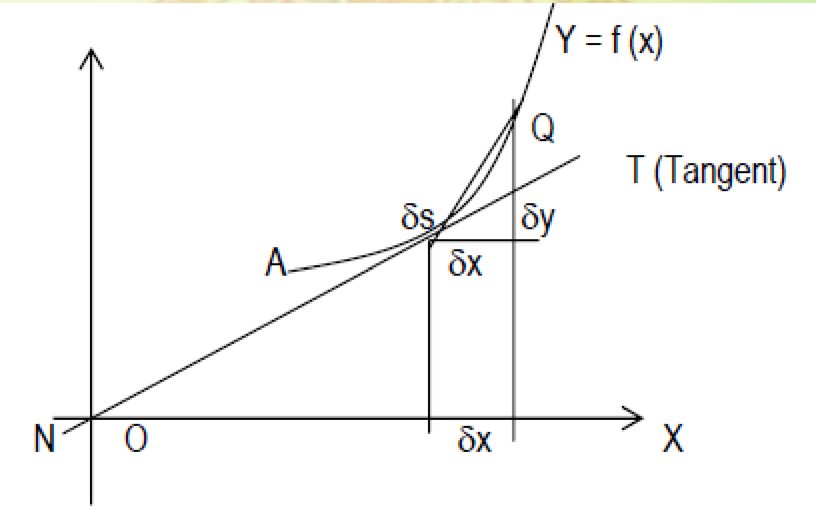
Relation (1) enables us to find the polar co-ordinates

(*r, θ) when the Cartesian co-ordinates (x, y) are known.*

Expression for arc length in Cartesian form.

Proof:

Let P (x,y) and Q (x + δx, Y + δy) be two neighboring points on the graph of the function y = f (x). So that they are at length S and S + δs measured from a fixed point A on the curve.



$$\therefore \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

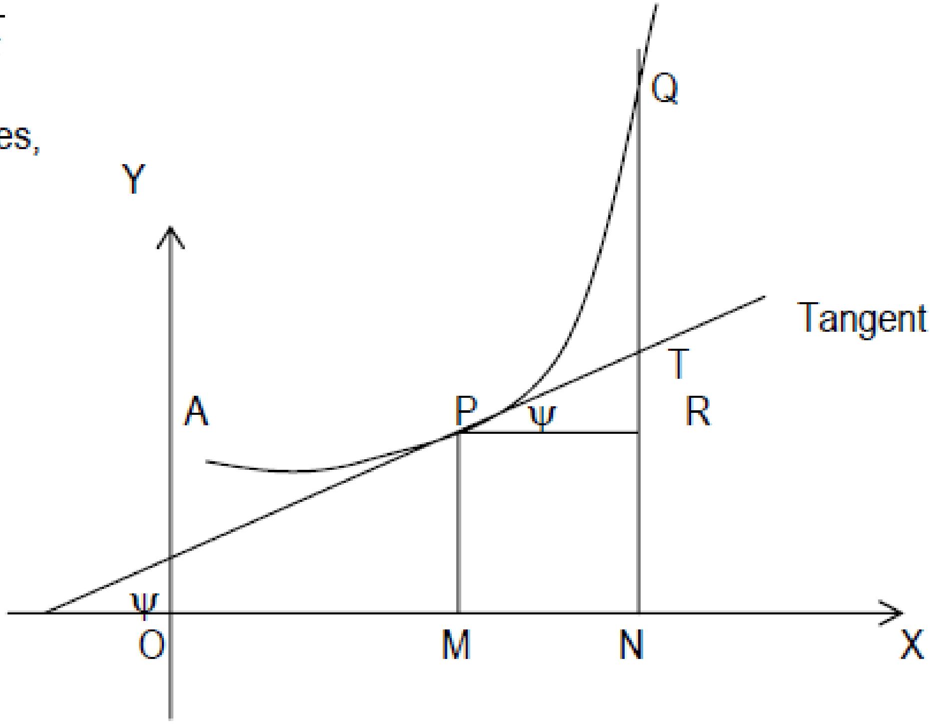
$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

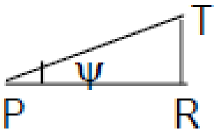
Expressions for $\frac{ds}{dx}$ & $\frac{ds}{dy}$

Trace a tangent to the curve at the point P, it makes an angle ψ with the x – axis. From Δ[le] PRT, we have

$$\tan\psi = \frac{dy}{dx}$$

Equation (2) becomes,





$$\frac{ds}{dx} = \sqrt{1+\tan^2\psi}$$

$$\frac{ds}{dx} = \sqrt{1 + \tan^2 \psi}$$

$$= \sqrt{Sec^2 \psi} = \sec \psi$$

$$\therefore \quad \frac{ds}{dx} = \text{Sec } \psi \text{ (Or) } \frac{dx}{ds} = \text{Cos } \psi$$

and equation (3) becomes

$$\frac{ds}{dy} = \sqrt{1 + Cot^2 \psi} = \sqrt{Co\sec^2 \psi} = \text{Cosec } \psi$$

$$\therefore \quad \frac{ds}{dy} = \text{Cosec}\psi \text{ or } \frac{dy}{ds} = \text{Sin } \psi$$

Derive an expression for arc length in parametric form.

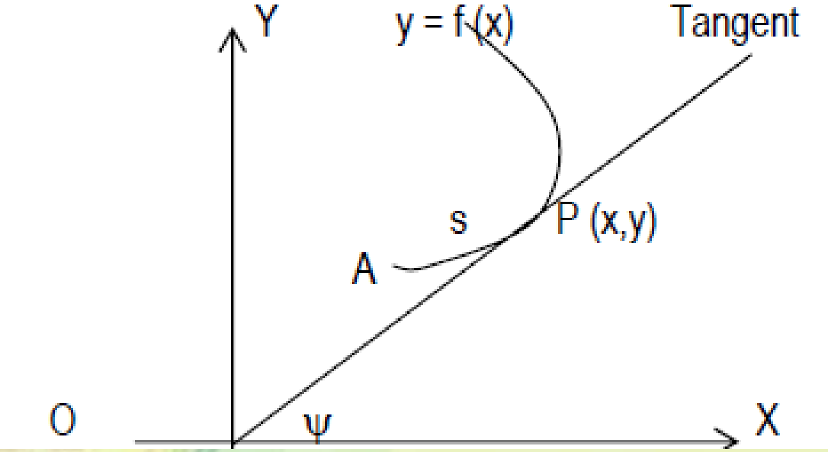
Solution: Let the equation of the curve in Parametric from be x = f (t) and y = g (t).

We have,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Curvature and Radius of curvature

- A curve cuts at every point on it. Which is determined by the tangent drawn.



Let P be a point on the curve y= f (x) at the length 's' from a fixed point A on it. Let the tangent at 'P' makes are angle ψ with positive direction of x – axis. As the point 'P' moves along curve, both s and ψ vary.

The rate of change ψ w.r.t s, i.e., $\frac{d\psi}{ds}$ as called the Curvature of the curve at 'P'.

The reciprocal of the Curvature at P is called the radius of curvature at P and is denoted by ρ.

$$\therefore \rho = \frac{1}{\frac{d\psi}{ds}} = \frac{ds}{d\psi}$$

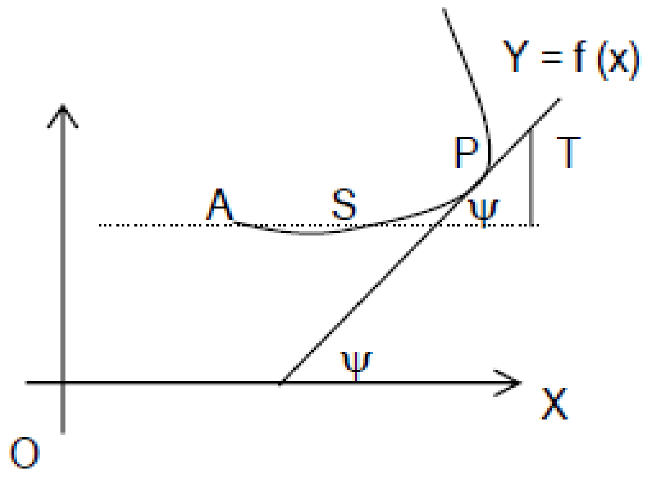
$$\therefore \rho = \frac{ds}{d\psi} \quad \text{(or)} \quad \frac{1}{\rho} = \frac{d\psi}{ds}$$

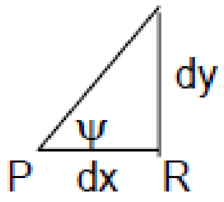
Also denoted $\rho = \frac{1}{K}$ $\quad \therefore K = \frac{d\psi}{ds}$

K is read it as Kappa.

Radius of curvature in Cartesian form

Solution :(1) Cartesian Form:





Let y = f (x) be the curve in Cartesian form.

We know that, $\tan\psi = \dfrac{dy}{dx}$ (From Figure) ----------- (1)

Where ψ is the angle made by the tangent at P with x – axis. Differentiating (1) W.r.t x, we get

$$= \frac{1}{\rho} \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$$

$$\therefore \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Where $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$

This is the formula for Radius of Curvature in Cartesian Form.

Radius of curvature in Polar form

- Let r = f(θ) be the curve in polar form. We know that, angle between radius vector and tangent

$$\text{Tan } \phi = r \frac{d\theta}{dr}$$

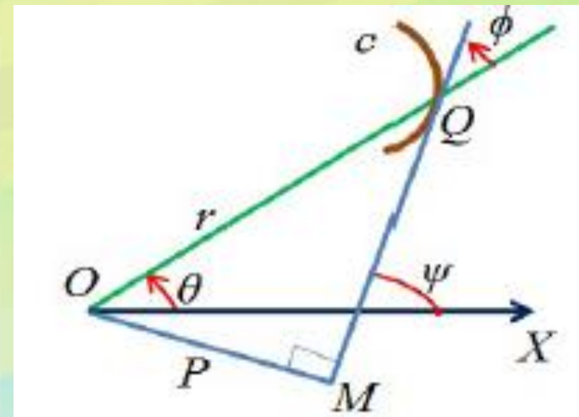
$$= r.\ \frac{1}{dr/d\theta}$$

i.e., $\tan \phi = \dfrac{r}{\left(dr/d\theta\right)}$

Differentiate w.r.t 'θ' we get

$$\text{Sec}^2 \phi \ .\ \frac{d\phi}{d\theta} = \frac{\frac{dr}{d\theta}\cdot\frac{dr}{d\theta} - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$\therefore \frac{d\phi}{d\theta} = \frac{1}{Sec^2\phi}\left\{\frac{\left(dr/d\theta\right)^2 r\, d^2r/d\theta^2}{\left(dr/d\theta\right)^2}\right\}$$



$$= \frac{1}{1+\tan^2\phi}\left\{\frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}\right\}$$

$$= \frac{1}{1+\frac{r^2}{\left(\frac{dr}{d\theta}\right)^2}}\left\{\frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}\right\}$$

$$\frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

From figure $\psi = \theta + \phi$

Differentiating w.r.t θ, we get

$$\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}$$

$$\therefore \frac{d\psi}{d\theta} = 1 + \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

$$\frac{d\psi}{d\theta} = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

Also we know that $\frac{ds}{d\theta} = \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}$

Now, $\rho = \frac{ds}{d\psi} = \frac{ds}{d\theta} \cdot \frac{d\theta}{d\psi}$

$$= \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}} \cdot \frac{\left(\frac{dr}{d\theta}\right)^2 + r^2}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$\rho = \frac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - rr_2} \text{ ---------- (3)}$$

Where $r_1 = \frac{dr}{d\theta}$, $r_2 = \frac{d^2r}{d\theta^2}$

Equation (3) as called the radius of curvature in Polar form.

Pedal Equation

Let $p = r \sin \phi$ be the curve in Polar Form.

We have $p = r \sin \phi$

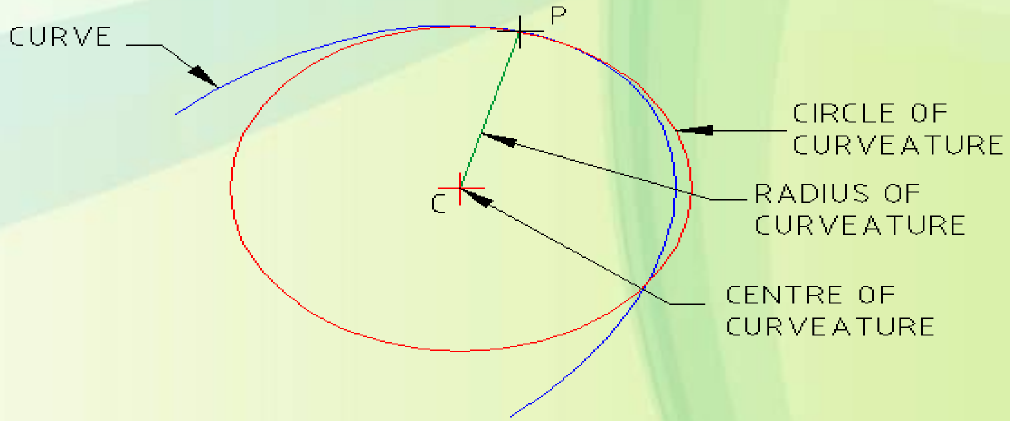
Differentiating p W.r.t r, we get

$$\rho = r \frac{dr}{dp}$$

is called Radius of Curvature of the Curve in Polar Form.

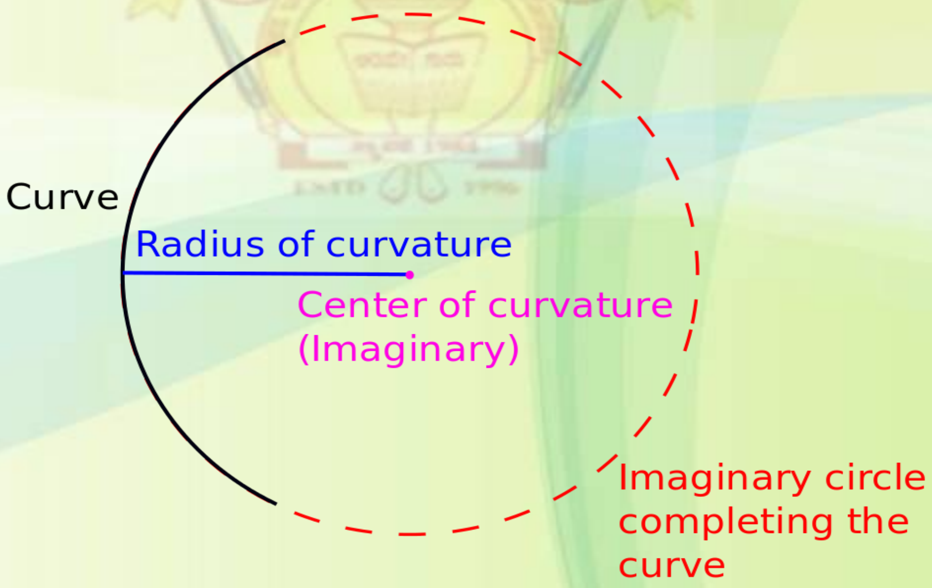
Centre and circle of curvature

The centre of curvature is the centre of the circle of curvature which is drawn through any point P on the curve so that the amount of curvature can be be measured at that point. The radius of curvature is shown in green and this is a normal to the curve at point P. The circle intersects the curve at point P. The evolute is the locus of the centres of curvature of any curve.



THE EVOLUTE OF AN ELLIPSE

The evolute is the locus of the centres of curvature of the ellipse, so find the centre of curvature for a number of points on one quadrant of the ellipse and sketch in the curve. Once you have the evolute for one quadrant, the rest can be found using axial symmetry through the major axis and minor axis.



circle of curvature

The circle with its center on the normal to the concave side of
a curve at a
given point on the curve and with its radius equal to the radius of
curvature at the point.

Queries ...?