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Department of Engg. Mathematics

Course : Calculus and Linear Algebra (18MAT11). Sem.: 1st (2018-19)

Course Coordinator:

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Differential Calculus - 2

Taylor's Theorem

Taylor's theorem states that if:

- $f(x)$ and its first $(n-1)$ derivatives are continuous in the $[a,b]$ and
- $f^n(x)$ exists in the (a,b)

then there exists at least one point $c \in (a,b)$ such that

$$f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(c)$$

Alternate form – If

- $f(x)$ and its first $(n-1)$ derivatives are continuous in the $[a, a+h]$ $(h>0)$ and
- $f^n(x)$ exists in the $(a, a+h)$

then there exists at least one point $\theta \in (0,1)$ such that $\langle (a+\theta h) \in (a, a+h) \rangle$

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

Taylor Series

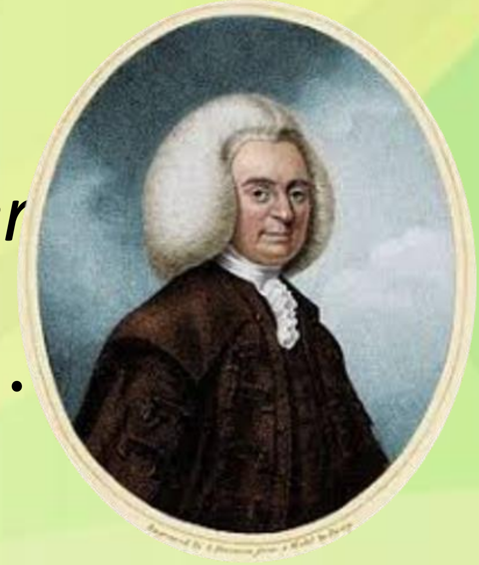
In the interval $[a, x]$, when $n \to \infty$ in Taylor's theorem we get Taylor series:

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ \ldots$$

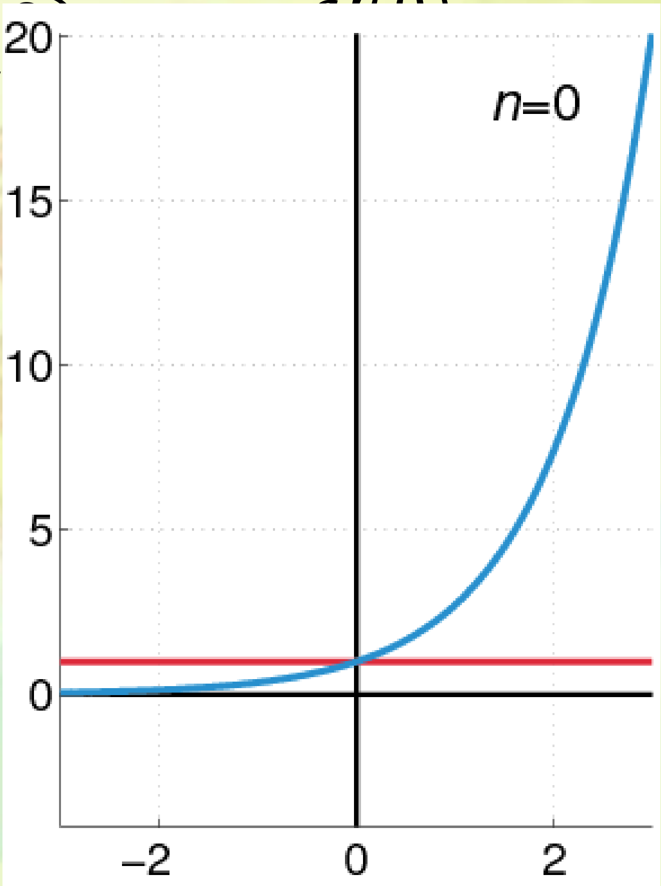
- The series is usedto create an approximation of what a function looks like

Maclaurin Series

Taylor series entered at zero is called *Maclaurin Ser*



$$f(x) = f(\quad x \quad x^2 \; f''^{(0)} + \cdots$$



Progression of Maclaurin Series approximation of the exponential function e^x based on n values.[1]

- The blue curve indicates the e^x function.
- The red curve indicates the sum of first $n + 1$ terms of the Maclaurin series.

Maclaurin Series – Calculating Limits

Find $\lim_{x\to 0}\left[\frac{x-\sin x}{x^n}\right]$, where n is an integer

$$\lim_{x\to 0}\left[\frac{x - \sin x}{x^n}\right] = \lim_{x\to 0}\left[\frac{x - \left(x - {x^3}/{3!} + {x^5}/{5!} \cdots\right)}{x^n}\right]$$

$$\cong \lim_{x\to 0}\left[\frac{{x^3}/{3!} - {x^5}/{5!} + \cdots}{x^n}\right]$$

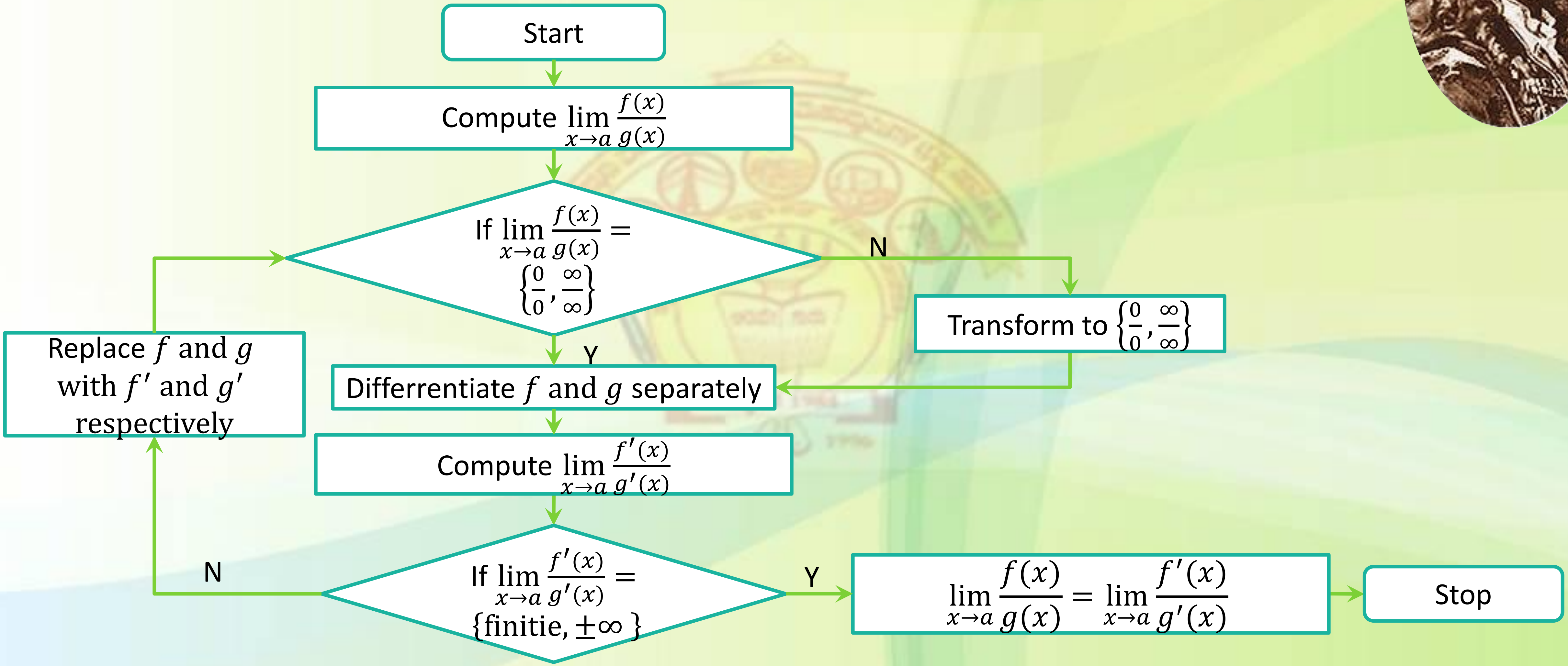
$$\cong \lim_{x\to 0}\left[\frac{x^{3-n}}{3!} - \frac{x^{5-n}}{5!} + \cdots\right]$$

Indeterminate Forms

- ***Indeterminate form*** : An expression that evaluates to

Form	Method
$^0/_0$	L'Hospital's Rule
$^\infty/_\infty$	L'Hospital's Rule
$0 \times \infty$	Change to $^0/_0$ or $^\infty/_\infty$ by moving one expression to the bottom
$\infty - \infty$	Change to $^0/_0$ or $^\infty/_\infty$ by using common denominator, rationalization, or factoring out a common factor
1^∞	Take $\log$ to make an indeterminate product
0^∞	Take $\log$ to make an indeterminate product
∞^0	Take $\log$ to make an indeterminate product

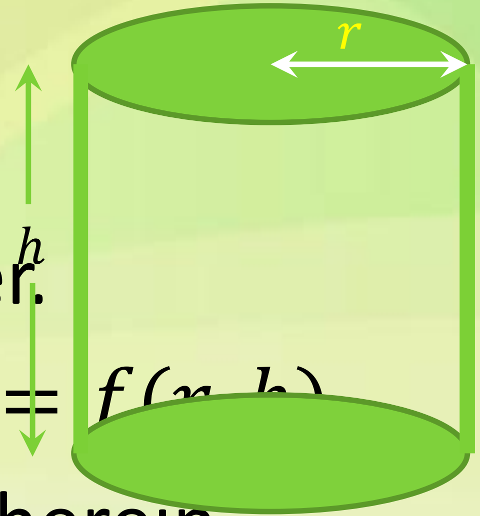
Applying L'Hospital's Rule



Partial Differentiation

- Volume of a cylinder:

$$V = \pi r^2 h$$



- Variables r and h are independent of each other.
- Volume V is dependent on both r and h , i.e.,$V = f(r, h)$
- $V = f(r, h)$ is a function of several variables, wherein
  - V is the dependent variable
  - r and h are independent variables
- Derivative of V is computed using *Partial Differentiation*

$$\text{Partial Derivative} = \frac{\partial(\text{Function of several variables})}{\ldots}$$

First Order Partial Derivative

- First order partial derivative of $u = f(x, y)$ with respect to independent variables:
  - $\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or u_x
  - $\frac{\partial u}{\partial y}$ or $\frac{\partial f}{\partial y}$ or u_y
- Similarly, the first order partial derivatives of $u = f(x, y, z)$ are:
  - $\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or u_x
  - $\frac{\partial u}{\partial y}$ or $\frac{\partial f}{\partial y}$ or u_y
  - $\frac{\partial u}{\partial z}$ or $\frac{\partial f}{\partial z}$ or u_z

Second Order Partial Derivative

$$\text{Second Order Partial Derivative} = \frac{\partial(\text{First Order Partial Derivative})}{\partial(\text{Independent variable})}$$

Second order partial derivative of $u = f(x, y)$ with respect to independent variables:

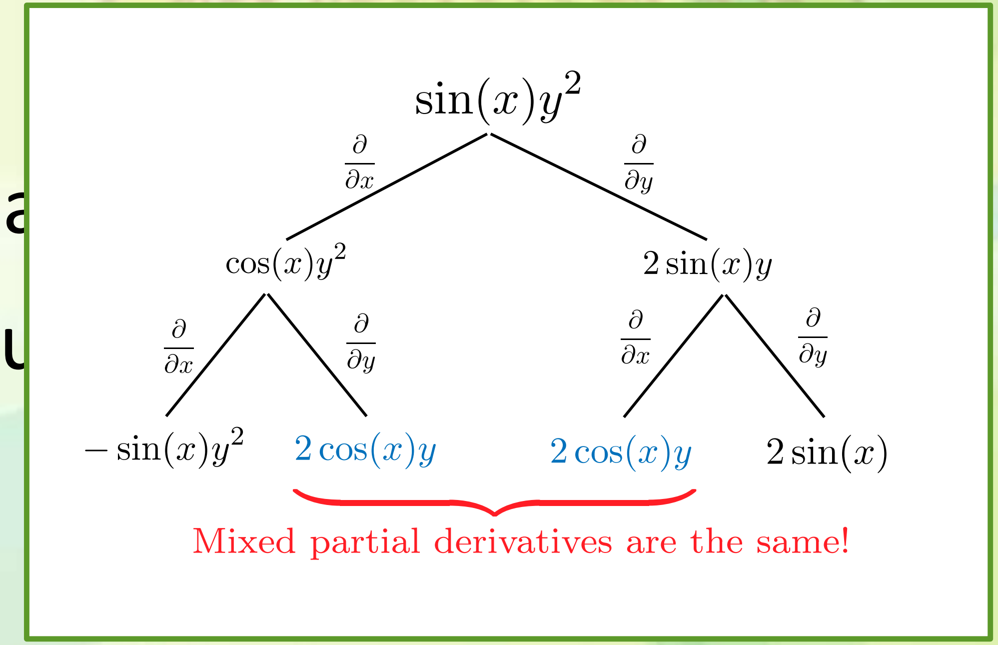
- $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$
- $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$
- $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$
- $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$

$$\text{Number of second order partial derivatives} = (\text{Number of independent variables in the given function})^2$$

Mixed Second Order Partial Derivative

Given $u = f(x, y)$

- $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$ are called *Mixed Second Order Partial Derivatives*
- $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, if a $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$ are continuous throu

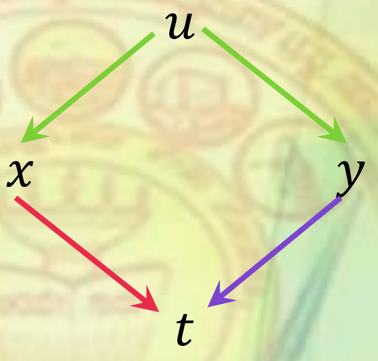


An example of computation of second order partial derivative.[2]

Total Derivative

$$u = f(x, y)$$

where $x = \emptyset(t)$y $= \varphi(t)$

- Variable dependency : 

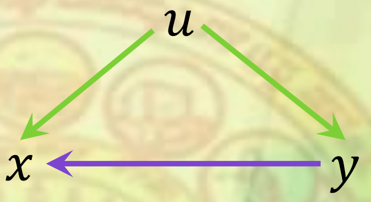
- Therefore t is the only independent variable.
- Derivative of u with respect to t:

$$\frac{du}{} = \frac{\partial u}{} \times \frac{dx}{} + \frac{\partial u}{} \times \frac{dy}{}$$

Implicit Function

Implicit Function: $u = f(x, y)$ and $u = 0$ or a constant

where $x = \emptyset(x)$ and $y = \varphi(x)$



- Variable dependency :
- x is the independent variable
- Since $u = 0$ or a constant, $\frac{du}{dx} = 0$
- From the definition of total derivative :

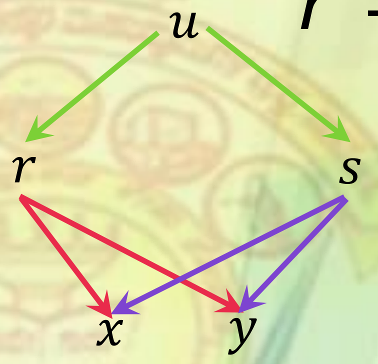
$$\frac{du}{} = \frac{\partial u}{} \times \frac{dx}{} + \frac{\partial u}{} \times \frac{dy}{} = 0$$

Composite Function

Composite Function: $u = f(r, s)$

where $r = \emptyset(x, y)$ s $= \varphi(x, y)$

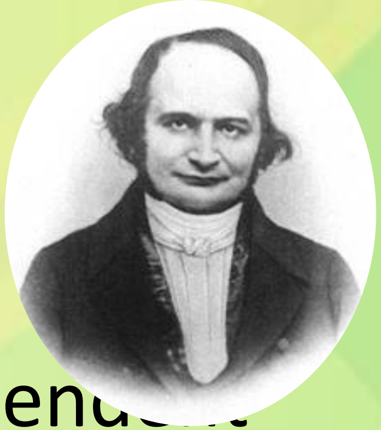
- Variable dependency :



- x and y are the two independent variables
- $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are computed as:

$\partial u \quad \partial u \quad \partial r \quad \partial u \quad \partial s \quad \partial u$

Jacobian



- $u = f(x, y)$ and $v = g(x, y)$, where x and y are independent variables

Jacobian of u and v w.r.t x and y $= J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} =$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

- $u = f(x, y, z), v = g(x, y, z)$ and $w = h(x, y, z)$, where

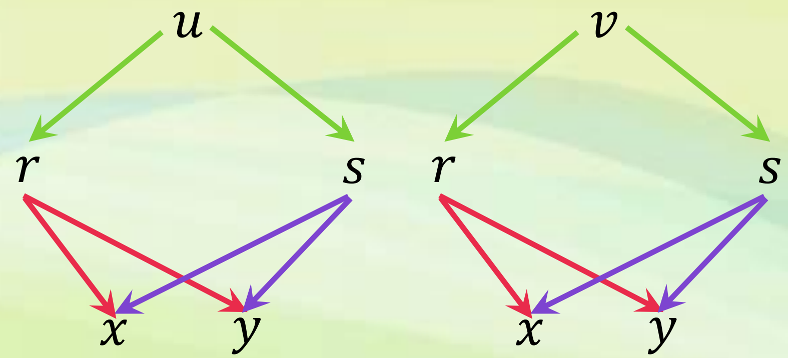
Jacobian Properties

1. $J = \frac{\partial(u,v)}{\partial(x,y)} J' = \frac{\partial(x,y)}{\partial(u,v)}$

then

$$J \times J' = 1$$

Composite functions



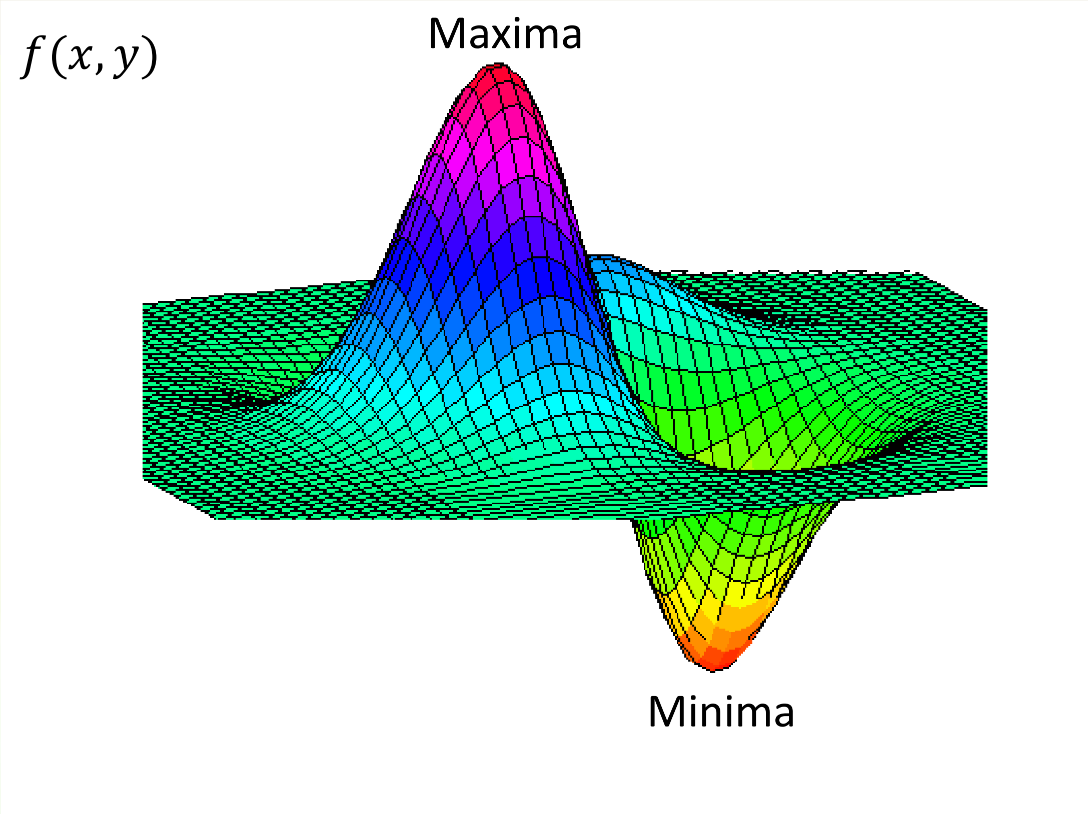
2. $u = f(r,s)$ and $v = g(r,s)$

where $r = \emptyset(x,y)$ $s = \varphi(x,y)$

then

$\partial(u,v)$ $\partial(u,v)$ $\partial(r,s)$

Maxima and Minima for $f(x, y)$



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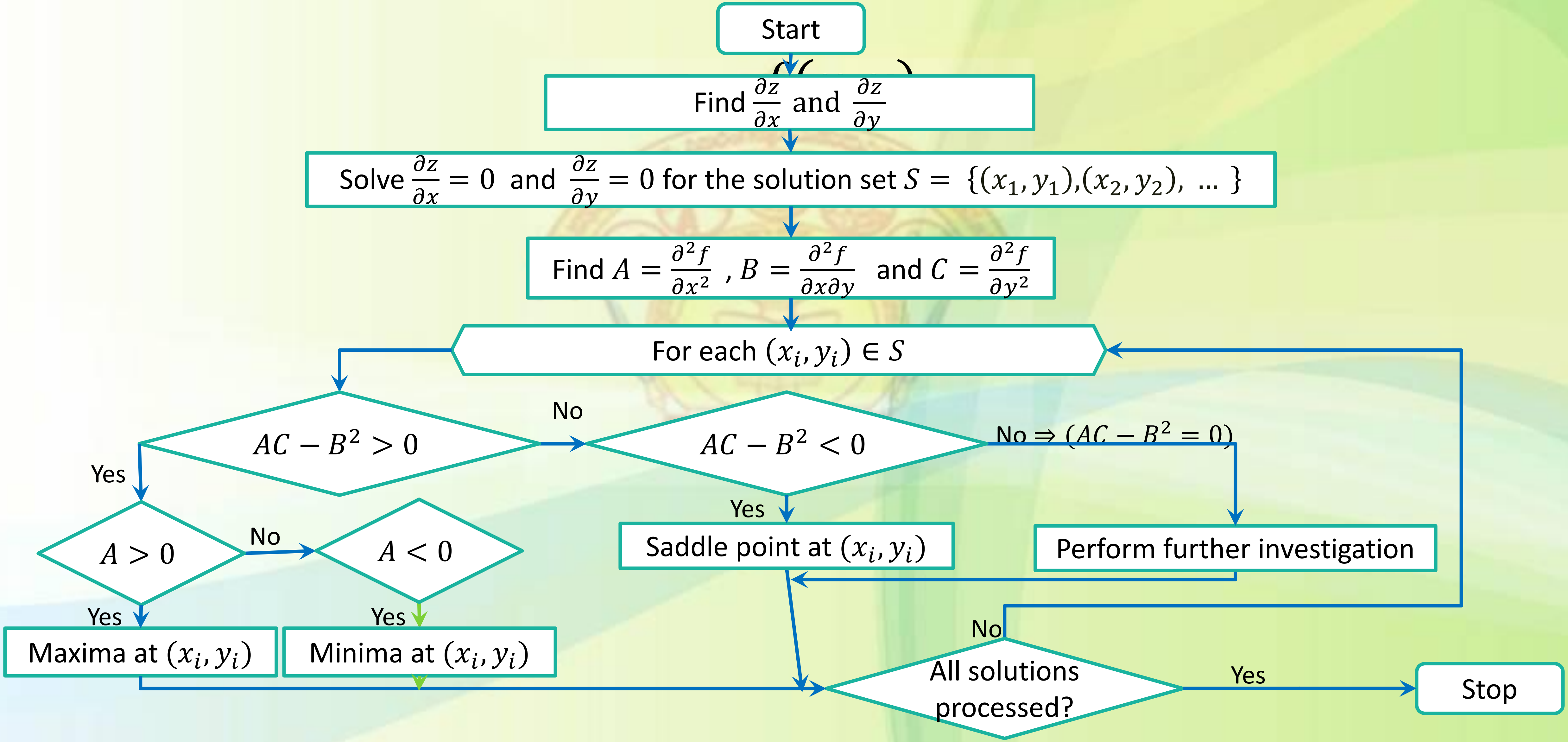
Maxima	Minima
$f(a, b) > f(a + h, b + k)$	$f(a, b) < f(a + h, b + k)$

Maxima and Minima – Working Rule



Maxima and Minima – Example

$$f(x,y) = 2x^2 - 4xy + y^4 + 2$$

$f_x = 4x - 4y$ $\qquad$ $f_y = -4x + 4y^3$

$f_x = 0$ $\qquad$ $f_y = 0$

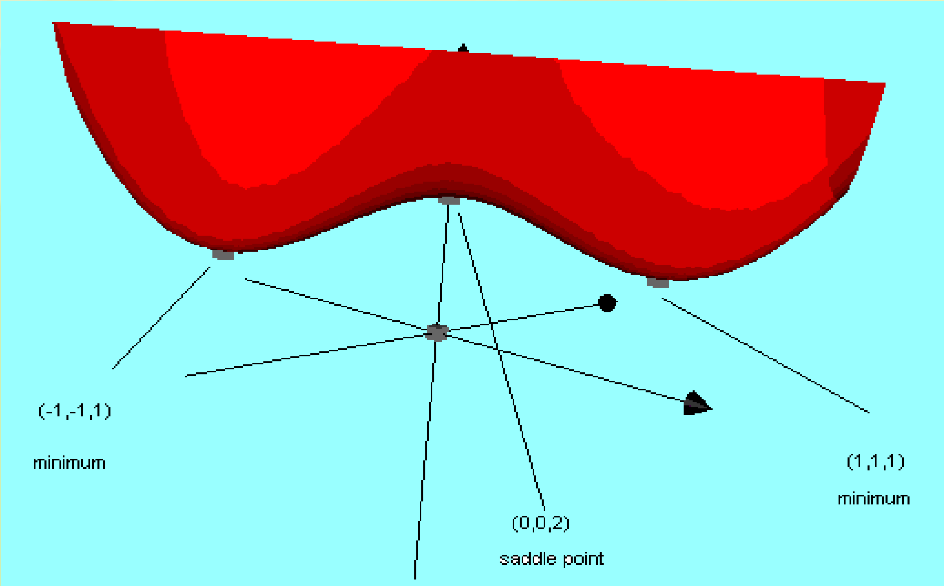
$4x - 4y = 0$ $\qquad$ $-4x + 4y^3 = 0$

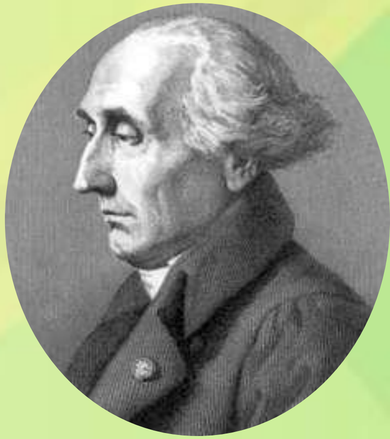
$\Rightarrow \quad x = y$ $\qquad \Rightarrow \quad y = 0,\ y = 1$ and $y = -1$

$\therefore$ The critical points are $(0,0),(1,1)$ and $(-1,-1)$

$A = f_{xx} = 4,\ B = f_{xy} = -4$ and $C = f_{yy} = 12y^2$

Critical points	$(0,0)$	$(1,1)$	$(-1,-1)$
$A = f_{xx} = 4$	$4 > 0$	$4 > 0$	$4 > 0$
$B = f_{xy} = -4$	-4	-4	-4
$C = f_{yy} = 12y^2$	0	12	12
$AC - B^2$	$-16 < 0$	$32 > 0$	$32 > 0$
Decision	Saddle point	Minimum	Minimum

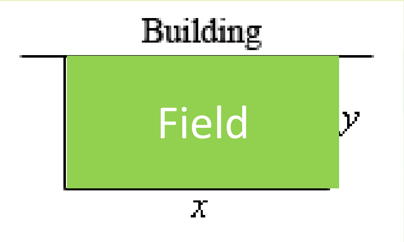




Lagrange's Undetermined Multipliers

- Enables to maximize or minimize a function that is subject to a constraint.

- Ex1: Designing the dimensions of a box to maximize its volume subject to a certain fixed amount of building material (and cost)



- Ex2: Maximize the area of a field subject to of fencing material.[4]

Maximize : $A = x * y$

Constraint: $500 = x + 2y$

Lagrange's – Working Rule

Step 1. Write auxiliary function $F(x, y, z) = f(x, y, z) + \lambda\, \emptyset(x, y, z)$, where λ is a parameter.

Step 2. Find F_x, F_y and F_z .

Step 3. Solve $F_x = 0$, $F_y = 0$ and $F_z = 0$ to compute λ and find the relations between x, y and z .

Step 4. Substitute the resulting relation in $\emptyset(x, y, z) = 0$ and solve for the solution set $S = \left\{ \begin{matrix} (x_1, y_1, z_1), \\ (x_2, y_2, z_2), \ldots \end{matrix} \right\}$ of stationary points.

Step 5. Find the corresponding maximum or minimum or both by substituting the stationary points in $f(x, y, z)$.

Queries ….?