# Department of Engg. Mathematics

**Course : Calculus and Linear Algebra (18MAT11).** **Sem.: 1st (2018-19)**

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# Integral Calculus

# Content

- Introduction
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- Triple integrals
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- Determination of volumes by multiple integrals
- Changing Order of integration
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# Introduction

The integral is an important concept in mathematics. The principles of integration were formulated independently by Isaac Newton and Gottfried Leibniz in the late 17th century. Through the fundamental theorem of calculus, which they independently developed, integration is connected with differentiation.

The multiple integral is a generalization of the definite integral to functions of more than one real variable, for example, f(x, y) or f(x, y, z). Integrals of a function of two variables over a region in R2 are called double integrals, and integrals of a function of three variables over a region of R3 are called triple integrals. Integral as area between two curves. Double integral as volume under a surface.

# Double integrals

The expression:

$$\int_{y=y_1}^{y_2}\int_{x=x_1}^{x_2} f(x,y)dx.dy$$

is called a *double integral* and indicates that $f(x, y)$ is first integrated with respect to $x$ and the result is then integrated with respect to $y$.

# Triple integrals

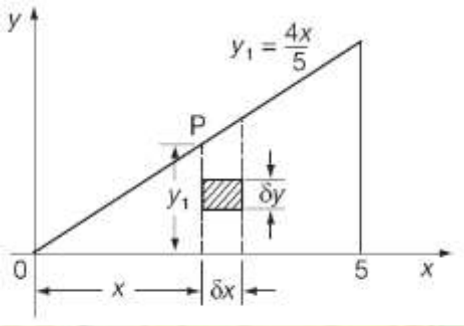
The expression:

$$\int_{z=z_1}^{z_2}\int_{y=y_1}^{y_2}\int_{x=x_1}^{x_2} f(x,y,z)\,dx.dy.dz$$

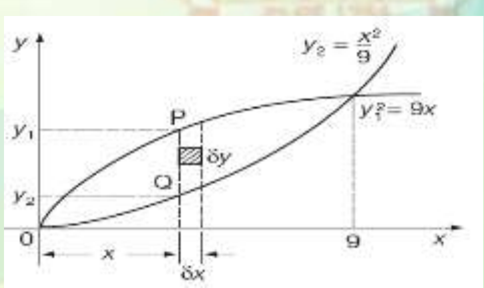
is called a *triple integral* and is evaluated by starting with the innermost integral and working outwards.

# Applications

Example 1: To find the area bounded by $y = \frac{4x}{5}$ the $x$-axis and the ordinate at $x = 5$.




Example 2: To find the area enclosed by the curves $y^2 = 9x$ and $y = \frac{x^2}{9}$




Example 3: Find the second moment of area of a rectangle 6 cm × 4 cm about an axis through one corner perpendicular to the plane of the figure.

## Alternative notation

Some times double integrals are written in a different way. For example, the integral:

$$\int_{x=0}^{6}\int_{y=0}^{4}(x^2+y^2)\,dy\,dx$$

could have been written as:

$$\int_{x=0}^{6}dx\int_{y=0}^{4}(x^2+y^2)\,dy$$

Here the working starts from the right-hand side integral.

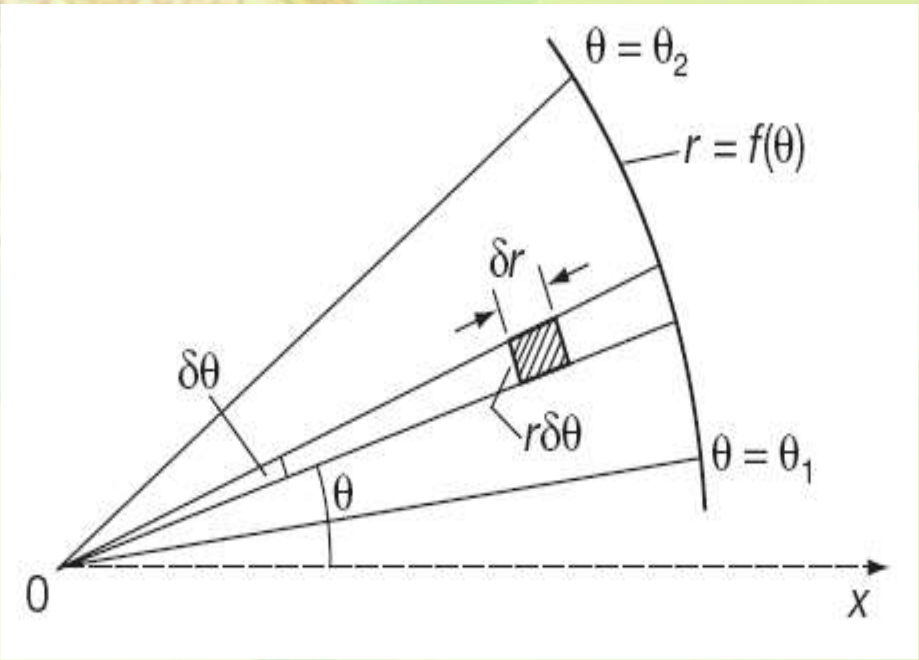
# Determination of areas by multiple integrals

1.To find the area of the polar curve $r = f(\theta)$ between the radius vectors $\theta = \theta_1$ and $\theta = \theta_2$ it is noted that the area of an element is $r.\delta r.\ \delta\theta$ .

$$A = \int_{\theta=\theta_1}^{\theta_2} \int_{r=0}^{r_1} r.dr.d\theta$$

$$= \int_{\theta=\theta_1}^{\theta_2} \left[ \frac{r^2}{2} \right]_0^{r_1} .d\theta$$

$$= \int_{\theta=\theta_1}^{\theta_2} \frac{1}{2} r_1^2 .d\theta$$



# Determination of volumes by multiple integrals

- The element of volume is:

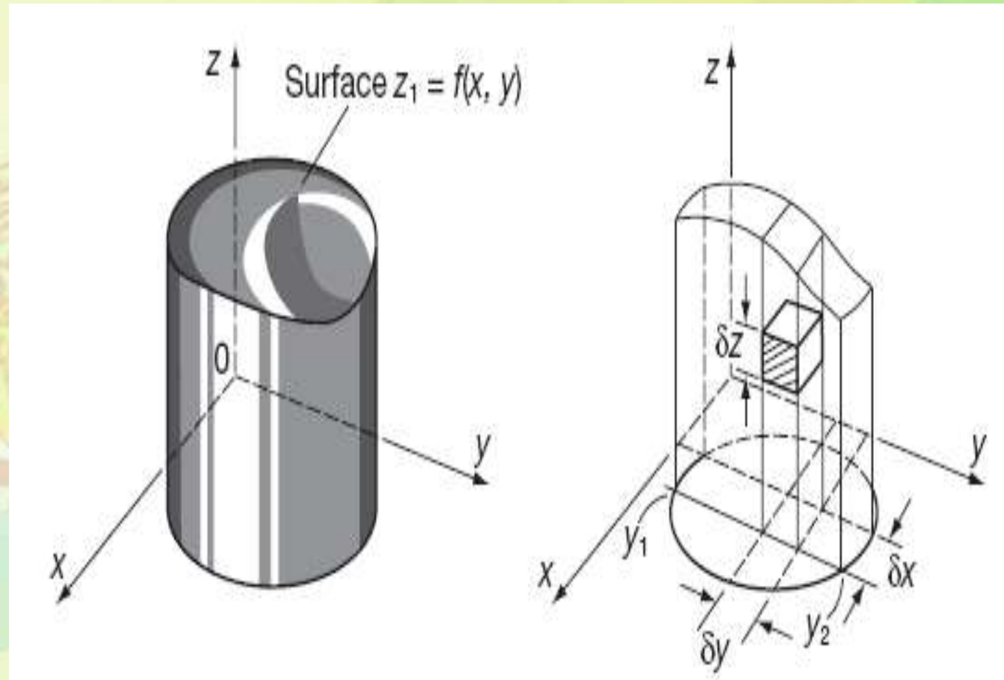
$$\delta V = \delta x . \delta y . \delta z$$

- Giving the volume $V$ as:

$$V \cong \sum_{x=x_1}^{x=x_2} \sum_{y=y_1}^{y=y_2} \sum_{z=z_1}^{z=z_2} dx.dy.dz$$

- That is:

$$V = \int_{x=x_1}^{x_2} \int_{y=y_1}^{y_2} \int_{z=z_1}^{z_2} dz.dy.dx$$



# Changing Order of integration

To evaluating a double integral we integrate first with respect to one variable and considering the other variable as constant, and then integrate with respect to the remaining variable. In the former case, limits of integration are determined in the given region by drawing stripes parallel to y-axis while in second case by drawing strips parallel to x-axis.

# Why do we need to study Integration?

Often we know the relationship involving the rate of change of two variables, but we may need to know the direct relationship between the two variables. For example, we may know the velocity of an object at a particular time, but we may want to know the position of the object at that time

## The processes of integration are used in many applications

- Historically, one of the first uses of integration was in finding the volumes of wine-casks (which have a curved surface).

- The PETRONAS Towers in Kuala Lumpur experience high forces due to winds. Integration was used to design the building for strength.

- The Sydney Opera House is a very unusual design based on slices out of a ball. Many differential equations were solved in the design of this building.

# Coordinate Conversion

$$\cos\theta = \frac{x}{r} \longrightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \longrightarrow y = r\sin\theta$$

$$\tan\theta = \frac{y}{x} \longrightarrow r^2 = x^2 + y^2$$ (Pythagorean Identity)

# Properties of Beta Function

$$B(x,y) = B(y,x).$$

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt, \qquad \mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0$$

$$B(x,y) = B(x,y+1) + B(x+1,y)$$

$$xB(x,y+1) = y\,B(x+1,y)$$

$$B(x,y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,d\theta, \qquad \mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0$$

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

$$B(x,y)\cdot B(x+y,1-y) = \frac{\pi}{x\sin(\pi y)},$$

# Queries …?