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Engg. Maths Dept.
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Department of Engg. Mathematics

Course : Calculus and Linear Algebra (18MAT11). **Sem.: 1st (2018-19)**

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Linear Algebra

Content

- Polar curves
- Angle between radius vector and tangent
- Angle between two curves
- Pedal equation
- Curvature and Radius of curvature
- Centre and circle of curvature
- The Evolutes of an ellipse

Rank of a Matrix in Echelon form

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

    4. The leading entry in each nonzero row is 1.
    5. Each leading 1 is the only nonzero entry in its column.

- An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form.)

- Any nonzero matrix may be **row reduced** (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique.

  Theorem 1: Uniqueness of the Reduced Echelon Form

  Each matrix is row equivalent to one and only one reduced echelon matrix.

PIVOT POSITION

- If a matrix *A* is row equivalent to an echelon matrix *U*, we call *U* **an echelon form** (or row echelon form) **of *A***; if *U* is in reduced echelon form, we call *U* **the reduced echelon form of *A***.

- A **pivot position** in a matrix *A* is a location in *A* that corresponds to a leading 1 in the reduced echelon form of *A*. A **pivot column** is a column of *A* that contains a pivot position.

Augmented Matrix

- We can write a system of linear equations as a matrix by writing only the coefficients and constants that appear in the equations.
  - This is called the augmented matrix of the system.

- Here is an example.

Linear System	Augmented Matrix
$\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x + 4z = 11 \end{cases}$	$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix}$

  – Notice that a missing variable in an equation corresponds to a 0 entry in the augmented matrix.

- Elementary row operations:

  1. Add a multiple of one row to another.
  2. Multiply a row by a nonzero constant.
  3. Interchange two rows.

  – Note that performing any of these operations on the augmented matrix of a system does not change its solution.

- Solve the system of linear equations.

$$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$$

  - Our goal is to eliminate the *x*-term from the second equation and the *x*- and *y*-terms from the third equation.

For comparison, we write both the system of equations and its augmented matrix.

System	Augmented Matrix
$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 1 & 2 & -2 & 10 \\ 3 & -1 & 5 & 14 \end{bmatrix}$
$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ 2y - 4z = 2 \end{cases}$	$\xrightarrow[R_3 - 3R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -4 & 2 \end{bmatrix}$

$$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ y - 2z = 1 \end{cases} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\begin{cases} x - y + 3z = 4 \\ z = 3 \\ y - 2z = 1 \end{cases} \xrightarrow{R_2 - 3R_3 \to R_2} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\begin{cases} x - y + 3z = 4 \\ y - 2z = 1 \\ z = 3 \end{cases} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- Now, we use back-substitution to find that:

$$x = 2, y = 7, z = 3$$

  - The solution is (2, 7, 3).

Gaussian Elimination

- A matrix is in row-echelon form if it satisfies the following conditions.
    1. The first nonzero number in each row (reading from left to right) is 1. This is called the leading entry.
    2. The leading entry in each row is to the right of the leading entry in the row immediately above it.
    3. All rows consisting entirely of zeros are at the bottom of the matrix.

- A matrix is in reduced row-echelon form if it is in row-echelon form and also satisfies the following condition.

4. Every number above and below each leading entry is a 0.

- To solve a system using Gaussian elimination, we use:

1. Augmented matrix
2. Row-echelon form
3. Back-substitution

1. Augmented matrix
   - Write the augmented matrix of the system.
2. Row-echelon form
   - Use elementary row operations to change the augmented matrix to row-echelon form.

3. Back-substitution

– Write the new system of equations that corresponds to the row-echelon form of the augmented matrix and solve by back-substitution.

- Solve the system of linear equations using Gaussian elimination.

$$\begin{cases} 4x + 8y - 4z = 4 \\ 3x + 8y + 5z = -11 \\ -2x + y + 12z = -17 \end{cases}$$

  – We first write the augmented matrix of the system.
  – Then, we use elementary row operations to put it
    in row-echelon form.

$$\xrightarrow{R_3 - 5R_2 \to R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & -10 & 20 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{10}R_3} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

- We now have an equivalent matrix in row-echelon form.
- The corresponding system of equations is:

  – We use back-substitution to solve the system.

The solution of the system is:

(–3, 1, –2)

Gauss-Jordan Elimination

- Use the elementary row operations to put the matrix in row-echelon form.
- Obtain zeros above each leading entry
by adding multiples of the row containing that entry to the rows above it.
- Begin with the last leading entry and work up.

$$\begin{bmatrix} 1 & \_ & \_ & \_ \\ 0 & 1 & \_ & \_ \\ 0 & 0 & 1 & \_ \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \_ & 0 & \_ \\ 0 & 1 & 0 & \_ \\ 0 & 0 & 1 & \_ \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \_ \\ 0 & 1 & 0 & \_ \\ 0 & 0 & 1 & \_ \end{bmatrix}$$

Using the reduced row-echelon form to solve a system is called Gauss-Jordan elimination.

Eigenvalues and Eigenvectors

- The vector **v** is an eigenvector of matrix *A* and **λ** is an eigenvalue of *A* if:

$$Av = \lambda v$$

(assume non-zero v)

- **Interpretation:** the linear transformation implied by *A* cannot change the direction of the eigenvectors v, only their magnitude.

Computing λ and v

- To find the eigenvalues λ of a matrix *A*, find the roots of the *characteristic polynomial*: $det(A - \lambda I) = 0$

$$A = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}$$

$$Av = \lambda v$$

$$det(\begin{bmatrix} 5-\lambda & -2 \\ 6 & -2-\lambda \end{bmatrix}) = 0 \text{ or } \lambda^2 - 3\lambda + 2 = 0 \text{ or } \lambda_1 = 1,\ \lambda_2 = 2$$

$$v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

Power method

The special advantage of the power method is that the eigenvector corresponds to the dominant eigenvalue and is generated at the same time. The inverse power method solves for the minimal eigenvalue/vector pair.

The disadvantage is that the method only supplies obtains one eigenvalue

Eigenvalues can be ordered in magnitude and the largest is called the dominant eigenvalue or spectral radius.

Think about how eigenvalues are a reflection of the nature of a matrix. Now if we multiply by that matrix over and over again..eventually the biggest eigenvalue will make everyone else have eigen-envy.

One λ to rule them all, One λ to find them, One λ to bring them all and in the darkness bind them.

Example of Power Method

Consider the follow matrix A

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Assume an arbitrary vector $x_0 = \{ 1\ 1\ 1\}^T$

Multiply the matrix by the matrix [A] by {x}

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 5 \\ -3 \\ -1 \end{Bmatrix} \qquad \Rightarrow \begin{Bmatrix} 5 \\ -3 \\ -1 \end{Bmatrix} = 5 \begin{Bmatrix} 1 \\ -0.6 \\ -0.2 \end{Bmatrix}$$

Normalize the result of the product

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.6 \\ -0.2 \end{Bmatrix} = \begin{Bmatrix} 4.6 \\ 1 \\ 0.2 \end{Bmatrix} \qquad \Rightarrow \begin{Bmatrix} 4.6 \\ 1 \\ 0.2 \end{Bmatrix} = 4.6 \begin{Bmatrix} 1 \\ 0.217 \\ 0.0435 \end{Bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.217 \\ 0.0435 \end{Bmatrix} = \begin{Bmatrix} 4.2174 \\ 0.4783 \\ -0.0435 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 4.2174 \\ 0.4783 \\ -0.0435 \end{Bmatrix} = 4.2174 \begin{Bmatrix} 1 \\ 0.1134 \\ -0.0183 \end{Bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.1134 \\ -0.0183 \end{Bmatrix} = \begin{Bmatrix} 4.1134 \\ 0.2165 \\ 0.0103 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 4.1134 \\ 0.2165 \\ 0.0103 \end{Bmatrix} = 4.1134 \begin{Bmatrix} 1 \\ 0.0526 \\ 0.0025 \end{Bmatrix}$$

As you continue to multiple each successive vector $\lambda = 4$ and the vector $u_k=\{1\ 0\ 0\}^T$

Queries ...?