Department of Engg. Mathematics

Course : Calculus and Linear Algebra (18MAT11). **Sem.: 1st (2018-19)**

Course Coordinator:

Prof. S. S. Thabaj

Ordinary Differential Equations of First Order

Content

- Exact Equation
- Bernoulli's Equations
- Orthogonal trajectories
- Newton’s Law of cooling
- LR circuit
- Singular solutions
- Clairaut’s equations

Differential Equations

An equation which involves unknown function and its derivatives

- ordinary differential equation (ode) : not involve partial derivatives
- partial differential equation (pde) : involves partial derivatives
- *order* of the differential equation is the order of the highest derivatives

Examples:

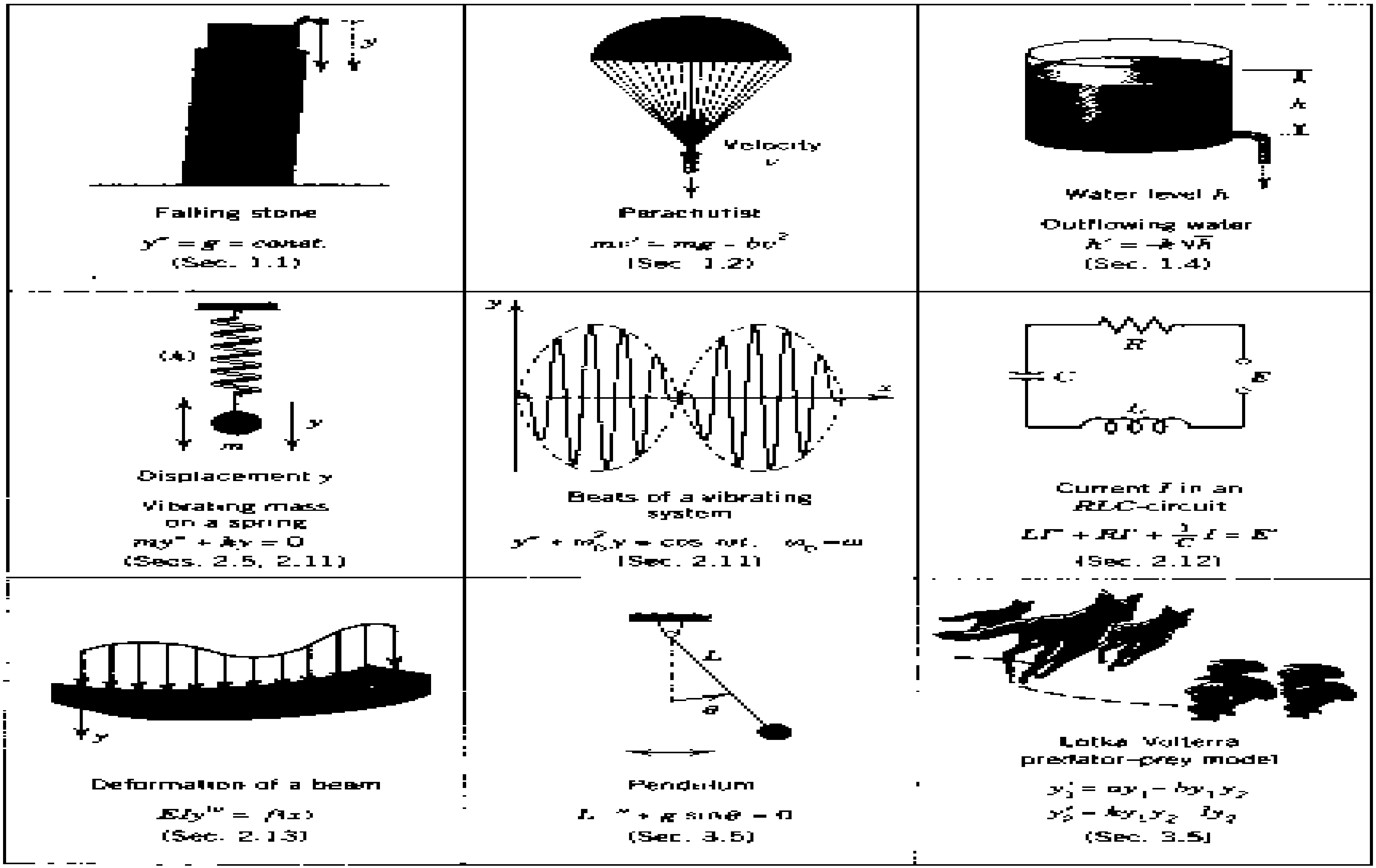
$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 3x\sin y$$

→ **second order ordinary differential equation**

$$\frac{\partial y}{\partial t} + x\frac{\partial y}{\partial x} = \frac{x+t}{x-t}$$

→ **first order partial differential equation**

Some Application of Differential Equation in Engineering



Definition:
A differential equation of type $M(x,y)dx+N(x,y)dy=0$ is called an exact differential equation

Test for Exactness

Let functions $M(x,y)$ and $N(x,y)$ have continuous partial derivatives in a certain domain D. The differential equation $M(x,y)dx+N(x,y)dy=0$ is an exact equation if and only if $\partial M/\partial x=\partial N/\partial y$.

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is an exact equation if :

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The solutions are given by the implicit equation

where : $\partial F/\partial x = M(x,y)$ and $\partial F/\partial y = N(x,y)$

$$F(x,y) = C$$

Non Exact Equation

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is a non exact equation if :

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The solutions are given by using integrating factor to change the equation into exact equation

Solution :

1. Check if : $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \Rightarrow \text{ function of } x \text{ only}$$

then integrating factor is $$e^{\int f(x)dx}$$

or if : $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y) \Rightarrow \text{ function of } y \text{ only}$$

then integrating factor is

2. Multiply the differential equation with integrating factor which result an exact differential equation

3. Solve the equation using procedure for an exact equation

Bernoulli Equation

A first-order ODE is said to be **linear** if it can be brought into the form

(1) $$y' + p(x)y = r(x),$$

by algebra, and **nonlinear** if it cannot be brought into this form.

Bernoulli’s equation

A Bernoulli equation is a differential equation of the form:

$$\frac{dy}{dx} + Py = Qy^n$$

This is solved by:

(a) Divide both sides by y^n to give:

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

(b) Let $z = y^{1-n}$ so that:

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

BERNOULL`S EQUATION

Integrating Factor

$$= e^{\int (1-n)p(x)\,dx}$$

Solution form of Bernoulli's equation

$$y^{1-n}\mathbf{e}^{\int (1-n)\,\mathbf{p}(\mathbf{x})\,\mathbf{d}x} = \int (1-n)Q\mathbf{e}^{\int (1-n)^{p}(x)\,\mathbf{d}x}\,\mathbf{d}x + c$$

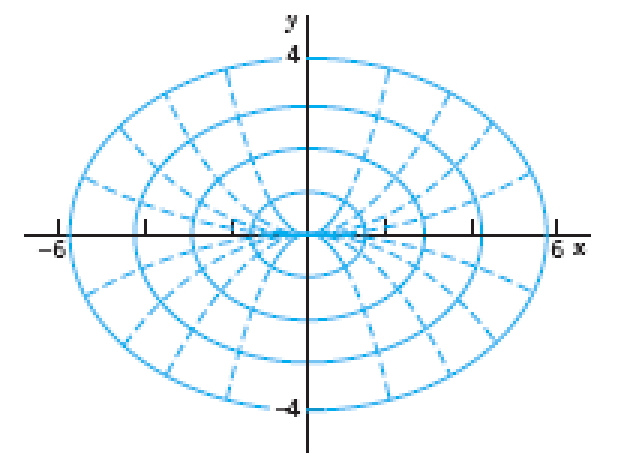
Orthogonal Trajectories

An important type of problem in physics or geometry is to find a family of curves that intersect a given family of curves at right angles. The new curves are called **orthogonal trajectories** of the given curves (and conversely). Examples are curves of equal temperature *(isotherms)* and curves **of heat** flow, curves of equal altitude *(contour lines)* on a map and curves of steepest descent on that map, curves of equal potential (*equipotential curves*, curves of equal voltage-the ellipses in Fig. 24, next slide) and curves of electric force (the parabolas in Fig. 24).

Here the **angle of intersection** between two curves is defined to be the angle between the tangents of the curves at the intersection point. *Orthogonal* is another word for *perpendicular.*

(continued)

In many cases orthogonal trajectories can be found using ODEs. In general, if we consider $G(x, y, c) = 0$ to be a given family of curves in the xy-plane, then each value of c gives a particular curve. Since c is one parameter, such a family is called a **one-parameter family of curves**.



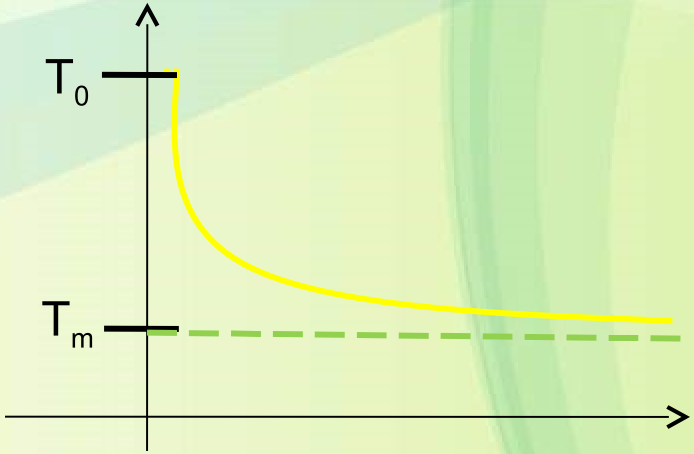
Electrostatic field between two ellipses (elliptic cylinders in space): Elliptic equipotential curves (equipotential surfaces) and orthogonal trajectories (parabolas)

Use Newton's Law of Cooling

- An object's temperature over time will approach the temperature of its surroundings (the medium)

- The greater the difference between the object's temperature and the medium's temperature, the greater the rate of change of the object's temperature

- This change is a form of exponential decay



The rate at which an object cools is proportional to the difference in temperature between the object and the surrounding medium:

$$\frac{dT}{dt} = k(T - M)$$

Where T is temperature of the object,

k is a proportionality constant,

M is the temperature of the surrounding medium

and t is time

A coroner uses this to help determine the time of death and is seen in every "Crime" TV series from Dragnet to CSI.

Given $$\frac{dT}{dt} = k(T - M)$$

by changing the variable T to $y(t) = T - M$ we get the following equation

$$\frac{dy}{dt} = k(y)$$

a very familiar differential equation, whose solution is

$$y(t) = y_0e^{kt}$$

changing back to T, we get $T(t) = T_m + (T_0 - T_m)e^{kt}$

where k will always be negative (from decay)

Example: A potato is taken out of a 300° F oven and left to cool in a room at 75° F. Write a differential equation expressing the change in rate of the temperature of the potato, *T*, with respect to time, *t*.

$$\frac{dT}{dt} = k(T - M)$$

$$\frac{dT}{dt} = k(300 - 75)$$

$$\frac{dT}{dt} = 225k$$

$$\frac{dT_o}{dt} = k(T_o - T_m)$$

$$T(t) = T_m + (T_o - T_m)e^{-kt}$$

$$T(t) = 75 + (300 - 75)e^{-kt}$$

$$T(t) = 75 + 225e^{-kt}$$

Newton's Law of Cooling

Example: The great detective Sherlock Holmes and his assistant, Dr. Watson, are discussing the murder of actor Cornelius McHam. McHam was shot in the head, and his understudy, Barry Moore, was found standing over the body with the murder weapon in hand. Let's listen in:

Watson: Open-and-shut case, Holmes. Moore is the murderer.

Holmes: Not so fast, Watson – you are forgetting Newton's Law of Cooling!

Watson: Huh?

Holmes: Elementary, my dear Watson. Moore was found standing over McHam at 10:06 p.m., at which time the coroner recorded a body temperature of 77.9°F and noted that the room thermostat was set to 72°F. At 11:06 p.m. the coroner took another reading and recorded a body temperature of 75.6°F. Since McHam's normal temperature was 98.6°F, and since Moore was on stage between 6:00 p.m. and 8:00 p.m., Moore is obviously innocent. Ask any calculus student to figure it out for you.

How did Holmes know that Moore was innocent?

Newton's Law of Cooling

$$H - H_s = (H_0 - H_S)e^{kt}$$

Example

A pizza pan is removed at 3:00 PM from an oven whose temperature is fixed at 450° F into a room that is a constant 70° F. After 5 minutes, the pizza pan is at 300° F. How long will it take for the pan to cool to 135° F?

Find k

$$300 - 70 = (450 - 70)e^{k5}$$

$$230 = 380e^{k5}$$

$$\frac{230}{380} = e^{k5}$$

$$ln\frac{23}{38} = 5k$$

$$k = \frac{ln\frac{23}{38}}{5} = -0.100418$$

t @ 135° F

$$135 - 70 = (450 - 70)e^{-0.100418t}$$

$$65 = 380e^{-0.100418t}$$

$$\frac{65}{380} = e^{-0.100418t}$$

$$ln\frac{13}{75} = -0.100418t$$

$$t = 17.45\ minutes$$

The Problem

Spencer and Vikalp are cranking out math problems at Safeway. Shankar is at home making pizza. He calls Spencer and tells him that he is taking the pizza out from the oven right now. Spencer and Vikalp need to get back home in time so that they can enjoy the pizza at a warm temperature of 110°F.

The pizza, heated to a temperature of 450°F, is taken out of an oven and placed in a 75°F room at time $t=0$ minutes. If the pizza cools from 450° to 370° in 1 minute, how much longer will it take for its temperature to decrease to 110°?

How to do it

$$\frac{dy}{dt} = k(y - 75)$$

$$(\tfrac{1}{y-75})dy = kdt$$

$$\int \tfrac{1}{y-75}\,dy = \int kdt$$

$$\ln|y - 75| = kt + C_1$$

$$y - 75 = e^{kt + C_1}$$

$$y = 75 + Ce^{kt}$$

$$450 = 75 + Ce^{k(0)}$$

$$375 = C$$

$$370 = 75 + 375e^{k(1)}$$

$$\tfrac{295}{375} = e^{k}$$

$$\ln(\tfrac{295}{375}) = k$$

$$110 = 75 + 375e^{(\ln\frac{295}{375})(t)}$$

$$\ln(\tfrac{35}{375}) = -0.23995t$$

$$t = 9.88363\,\text{min}$$

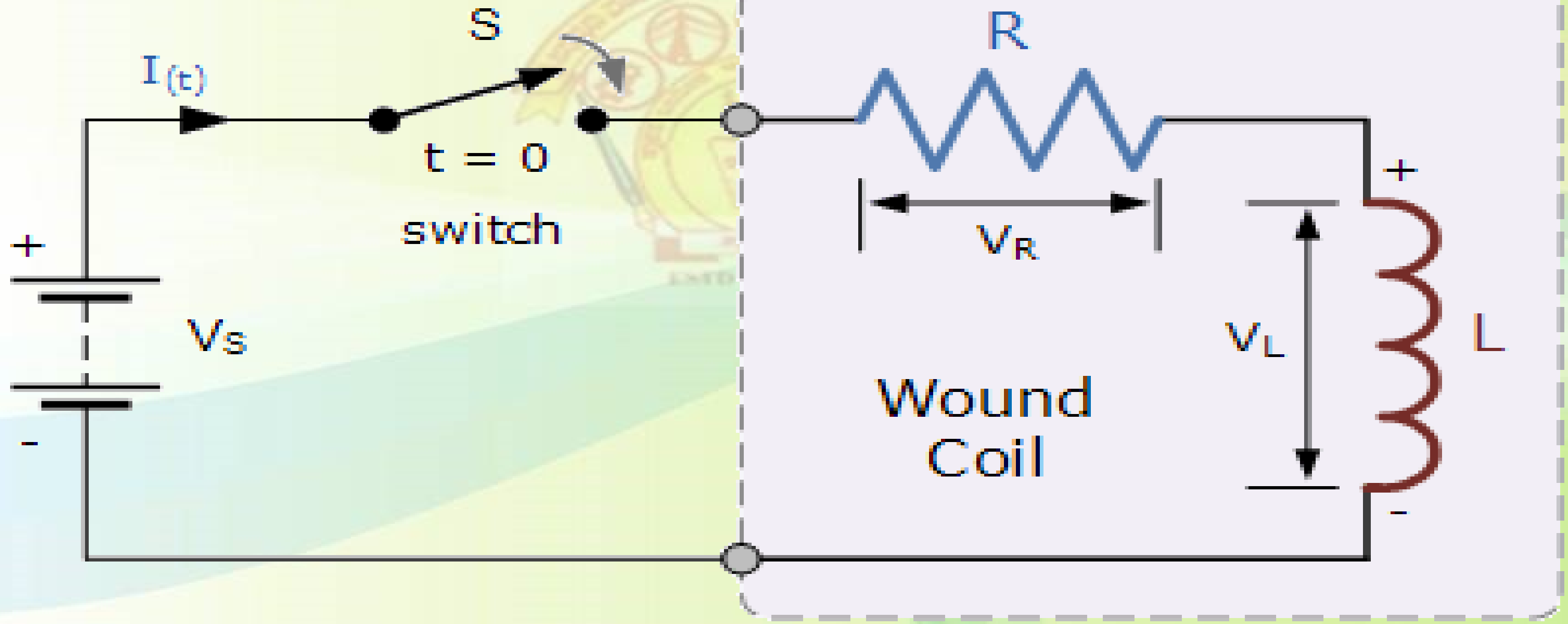
It takes about 8.88363 more minutes
For the object to cool to a temperature of
110°

Real Life Applications

- To predict how long it takes for a hot cup of tea to cool down to a certain temperature
- To find the temperature of a soda placed in a refrigerator by a certain amount of time.
- In crime scenes, Newton's law of cooling can indicate the time of death given the probable body temperature at time of death and the current body temperature

LR Series Circuit

All coils, inductors, chokes and transformers create a magnetic field around themselves consist of an Inductance in series with a Resistance forming an LR Series Circuit.



R-L Circuits?
What does the "L" stand for?

Good Question! "L" stands for the self-inductance of an inductor measured in Henrys (H).

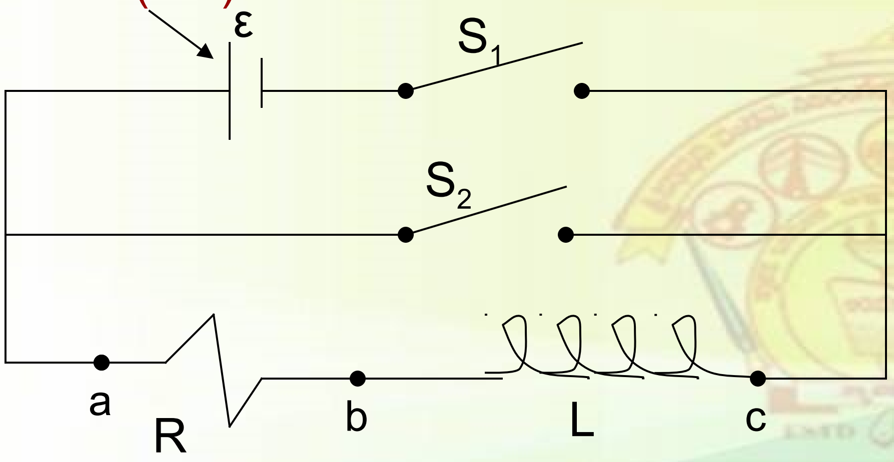
So…What is an inductor?

- **An inductor is an electronic device that is put into a circuit to prevent rapid changes in current.**
- **It is basically a coil of wire which uses the basic principles of electromagnetism and Lenz's Law to store magnetic energy within the circuit for the purposes of stabilizing the current in that circuit.**
- **The voltage drop across an inductor depends on the inductance value L and the rate of change of the current di/dt.**

$$V = L\frac{di}{dt}$$

R-L Circuits
The set up and initial conditions

Assume an ideal source (r=0)



An R-L circuit is any circuit that contains both a resistor and an inductor.

At time t = 0, we will close switch S1 to create a series circuit that includes the battery. The current will grow to a “steady-state” constant value at which the device will operate until powered off (i.e. the battery is removed)

Initial conditions: At time t = 0…when S_1 is closed…i = 0 and

$$\left(\frac{di}{dt}\right)_{initial} = \frac{\varepsilon}{L}$$

Singular solution

A singular solution $y_s(x)$ of an ordinary differential equation is a solution that is singular or one for which the initial value problem (also called the Cauchy problem by some authors) fails to have a unique solution at some point on the solution.

Definition of general solution. 1 : a solution of an ordinary differential equation of order n that involves exactly n essential arbitrary constants — called also complete solution, general integral. 2 : a solution of a partial differential equation that involves arbitrary functions — called also general integral.

Clairaut's equation

Clairaut's differential **equation**. **Clairaut's equation**, in mathematics, a differential **equation** of the form y = x (dy/dx) + f(dy/dx) where f(dy/dx) is a function of dy/dx only. The **equation** is named for the 18th-century French mathematician and physicist Alexis-Claude **Clairaut**, who devised it.

Queries ....?